Please provide complete and well-written solutions to the following exercises.

Due May 28, in the discussion section.

Assignment 7

Exercise 1. Let A be an $m\times n$ matrix. Let B be an $\ell\times m$ matrix. Show that $(BA)^t = A^t B^t$.

Exercise 2. Let $n \in \mathbf{N}$. Let S_n denote the set of permutations on $\{1,\ldots,n\}$. For any $\sigma \in S_n$, define $\mathrm{sign}(\sigma) := (-1)^N$, where σ can be written as the product of N transpositions.

Now, let A be an $n\times n$ matrix with entries A_{ij}, $i,j \in \{1,\ldots,n\}$. Consider the expression

$$F(A) := \sum_{\sigma\in S_n} \mathrm{sign}(\sigma) \prod_{i=1}^{n} A_{i\sigma(i)}.$$

- (a) Let A be a 2×2 matrix. Show directly that $F(A) = \det(A)$. (Hint: there are only two elements in S_2. What are they?)
- (b) Show that for any $n\times n$ matrix, $F(A) = \det(A)$.

Exercise 3. Let A denote the following matrix.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 5 & 0 \end{pmatrix}.$$

Compute $\det(A)$. Explain what formula you are using, and why your computation of $\det(A)$ is correct. (Hint: use the previous exercise.)

Exercise 4. Let $\mathbf{F}$ be a field, and let M be an $n\times n$ matrix with entries in the field $\mathbf{F}$. Suppose there exist matrices A, B such that A is a square matrix, and such that M can be written as

$$M = \begin{pmatrix} A & B \\ 0 & I \end{pmatrix}.$$

Show that $\det(M) = \det(A)$.

Exercise 5. Let $\mathbf{F}$ be a field, and let M be an $n\times n$ matrix with entries in the field $\mathbf{F}$. Suppose there exist matrices A, B, C such that A is a square matrix, and such that M can be written as

$$M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}.$$

Show that $\det(M) = \det(A)\cdot\det(C)$. (Hint: consider the matrix product $\begin{pmatrix} I & 0 \\ 0 & C \end{pmatrix}\begin{pmatrix} A & B \\ 0 & I \end{pmatrix}$, and apply the previous exercise to the result.)

Exercise 6. Define

$$A := \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}.$$

- Find all of the eigenvalues of A.
- For each eigenvalue λ of A, find the set of eigenvectors corresponding to λ.
- Find a basis for $\mathbf{R}^2$ consisting of eigenvectors of A (if possible).
- If you can find a basis of $\mathbf{R}^2$ consisting of eigenvectors of A, then find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

Exercise 7. Section 5.1, Exercise 8 in the textbook. (The phrase "T is a linear operator on a vector space V" means that $T\colon V \to V$ is a linear transformation.)

Exercise 8. Let T be a linear transformation on a vector space V, and let x be an eigenvector of T corresponding to the eigenvalue λ. For any positive integer m, prove that x is an eigenvector of T^m corresponding to the eigenvalue λ^m. Then, state and prove an analogous result for matrices.

Exercise 9. Define $T\colon M_{n\times n}(\mathbf{R}) \to M_{n\times n}(\mathbf{R})$ by $T(A) := A^t$. Note that T is a linear transformation.

- Show that ± 1 are the only eigenvalues of T.
- Describe the eigenvectors corresponding to each eigenvalue of T.
- Find an ordered basis β for $M_{2\times 2}(\mathbf{R})$ such that $[T]_\beta^\beta$ is a diagonal matrix.
- Find an ordered basis β for $M_{n\times n}(\mathbf{R})$ such that $[T]_\beta^\beta$ is a diagonal matrix for $n > 2$.

Exercise 10. Section 5.2, Exercise 2(ab) in the textbook.

Exercise 11. Consider the following 2×2 matrix.

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}.$$

Let n be an arbitrary positive integer. Find an expression for A^n.