Please provide complete and well-written solutions to the following exercises.

Due April 30, in the discussion section.

Assignment 4

Exercise 1. Section 2.2, Exercise 5(cd) in the textbook.

Exercise 2. Section 2.3, Exercise 4(cd) in the textbook.

Exercise 3. Section 2.3, Exercise 1(aefgh) in the textbook.

Exercise 4. Let V be a vector space over a field $\mathbf{F}$. Let $T\colon V \to V$ be a linear transformation.

(a) Prove that $T^2 = 0$ if and only if $R(T) \subseteq N(T)$. (Here we denote T^2 as the linear transformation such that $T^2(v) = T(T(v))$ for all $v \in V$.)
(b) Assume that V is finite-dimensional and $T^2 = 0$. Prove that $\text{nullity}(T) \geq \dim(V)/2$.

Exercise 5. Let $f(x)$ be a polynomial of degree n in one real variable x. Prove that the $n+1$ polynomials $f(x), f'(x), f''(x), \ldots, f^{(n)}(x)$ are linearly independent. Conclude that they span $P_n(\mathbf{R})$.

Exercise 6. Section 2.4, Exercise 1(bcdefhi) in the textbook.

Exercise 7. Let V, W be finite-dimensional vector spaces and let $T\colon V \to W$ be an isomorphism. Let X be a subspace of V.

- Show that $T(X)$ is a subspace of W.
- Show that $\dim(X) = \dim(T(X))$.

Exercise 8. Let V_1, V_2 each be three-dimensional subspaces of $\mathbf{R}^5$. Prove that V_1 and V_2 have a common nonzero vector.

Exercise 9. Prove the following assertions.

(a) Let A, B be $n \times n$ matrices. Then AB is invertible if and only if A and B are invertible.
(b) For arbitrary matrices A and B, it can occur that A, B are not invertible, whereas AB is invertible.