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MTHED-UE-1049: Mathematical Proof and Proving (MPP)
MATH-UA-125: Introduction to Mathematical Proofs

Homework No. 7

This homework should be submitted just before the beginning of class, on April 2nd, 2012. You should bring to class a copy of the homework that you submit, in order to participate in class discussion.

1. We proved in class that $\sqrt{2}$ is an irrational number. The proof was an indirect proof by contradiction. It starts by assuming that $\sqrt{2} = \frac{n}{m}$, where $\frac{n}{m}$ is a reduced fraction (i.e., n and m are integers that are relatively prime, $m \neq 0$).

   Explain in your own words why we made this assumption, and why it is not enough to assume that n and m are integers for which $m \neq 0$.

2. Prove that $\sqrt{3}$ is an irrational number.

3. Prove that there is no smallest positive real number.

4. Here are two definitions regarding a (binary) operation[1]:

   Let $\boldsymbol{S}$ be a set of elements, and $*$ an operation that is defined for any two elements of $\boldsymbol{S}$.

   i. The operation $*$ is called **commutative** if for all $a, b \in \boldsymbol{S}$, $a * b = b * a$.

   ii. The operation $*$ is called **associative** if for all $a, b \in \boldsymbol{S}$, $(a * b) * c = a * (b * c)$.

   (a) Give an example of a set ***S*** and an operation on ***S*** that is both commutative and associative.

   (b) Give an example of a set ***S*** and an operation on ***S*** that is neither commutative nor associative.

   (c) Can you think of an example of an operation that is commutative but not associative? Explain.

   (d) Can you think of an example of an operation that is associative but not commutative? Explain.

   (e) Here are a few operations. For each one, you need to determine whether it is commutative and whether it is associative. Prove all your claims:

   (e.1) $a * b = \frac{a+b}{2}$ for $a, b \in R$.

   (e.2) $a * b = a^b$ for $a, b \in N$.

   (e.3) $a * b = b$ for $a, b \in Q$.

[1] A (binary) operation on a (nonempty) set S maps each ordered pair of elements of S to one and only one element of S. In other words, given $a, b \in S$, an operation $*$ is defined so that $a * b \in S$. For example, division is a (binary) operation on the set of all non-zero rational numbers. However, it is not a (binary) operation on the set of integers. Can you explain why?