Please provide complete and well-written solutions to the following exercises.

Due February 6, at the beginning of class.

Assignment 5

Exercise 1. Let $s(t) = (x(t), y(t), z(t))$ be a continuously differentiable parametrization of a curve γ in Euclidean space $\mathbf{R}^3$, where $t \in [0, T]$ for some $T > 0$. This same curve can be parameterized by $r(t) = (x(t^2), y(t^2), z(t^2))$ where $t \in [0, \sqrt{T}]$. Let $f\colon \mathbf{R}^3 \to \mathbf{R}$ be a function. Show that $\int_\gamma f\, ds = \int_\gamma f\, dr$. That is, the value of the integral of f on γ does not depend on the parametrization. However, this fact is not necessarily true for line integrals of vector fields!

Exercise 2. Find the line integral of the function $f(x, y, z) = x + y + z$ over the straight line segment from $(1, 2, 3)$ to $(0, -1, 1)$.

Exercise 3. Let C be the curve in the plane that lies in the set $x^2 + y^2 = 4$ with $x \geq 0$ and $y \geq 0$, and so that C has endpoints $(2, 0)$ and $(0, 2)$. Compute the line integral of $f(x, y) = x^2 - y$ over C

Exercise 4. Sketch the following vector fields F in the plane by considering $-3 \leq x \leq 3$ and $-3 \leq y \leq 3$.

- $F(x, y) = (0, x)$
- $F(x, y) = \left(\dfrac{x}{x^2 + y^2}, \dfrac{y}{x^2 + y^2}\right)$
- $F(x, y) = \left(\dfrac{-y}{x^2 + y^2}, \dfrac{x}{x^2 + y^2}\right)$

Exercise 5.

- Let $F(x, y) = (x, y)$ be a vector field in the plane. Find a function $G\colon \mathbf{R}^2 \to \mathbf{R}$ such that $\nabla G = F$.
- Let $F(x, y, z) = (yz^2, xz^2, 2xyz)$ be a vector field on Euclidean space $\mathbf{R}^3$. Find a function $G\colon \mathbf{R}^3 \to \mathbf{R}$ such that $\nabla G = F$.

Exercise 6. Let $f\colon \mathbf{R} \to \mathbf{R}$, $g\colon \mathbf{R} \to \mathbf{R}$, $h\colon \mathbf{R} \to \mathbf{R}$ be continuous functions. Define the vector field

$$F(x, y, z) = (f(x), g(y), h(z)).$$

Find a function $G\colon \mathbf{R}^3 \to \mathbf{R}$ such that $F = \nabla G$. (Hint: consider the function $G(x, y, z) = \int_0^x f(t)dt$. What is ∇G? Can you modify this example so that $\nabla G = F$?)

Exercise 7. Let $s(t) = (3 + 5t^2, 3 - t^2, t)$ be a parametrization of the curve C, where $0 \leq t \leq 2$. Let $F(x, y, z) = (z^2, x, y)$ be a vector field. Compute the line integral $\int_C F \cdot T\, ds$.

Exercise 8. Let $s(t)$ be a parametrization of the curve C which travels in a straight line from $(0,0,0)$ to $(1,4,4)$. Let $F(x,y,z) = (x-y, y-z, z)$ be a vector field on Euclidean space $\mathbf{R}^3$. Compute the line integral $\int_C F \cdot T\, ds$. Compute also the line integral $\int_C F \cdot T\, dr$, where $r(t)$ is a parametrization of the straight line from $(1,4,4)$ to $(0,0,0)$.

Exercise 9. Let $F(x,y) = (x+y, -x^2 - y^2)$ be a vector field in the plane. Let C denote the triangle with vertices $(1,0)$, $(0,1)$ and $(-1,0)$. Compute the flux of the vector field F that is emanating outward across the triangle C.

Exercise 10. Let $F(x,y) = (x^2, y^2)$ be a vector field in the plane. Let C be the straight line segment between $(3,0)$ and $(0,3)$. Compute the flux of the vector field F moving upwards across C.

Exercise 11. Let $f\colon \mathbf{R}^3 \to \mathbf{R}$ be a function on Euclidean space $\mathbf{R}^3$. Let C be a curve in Euclidean space, which is parametrized by a function s. Assume that $f(x,y,z) \geq p$ for some real number p, for all points (x,y,z) in C. Define

$$I = \int_C f\, ds$$

Which of the following things is true? Explain your reasoning.

- $I \geq p$.
- $I \geq pL$, where L is the length of C.