Please provide complete and well-written solutions to the following exercises.

Due January 23, at the beginning of class.

Assignment 3

Exercise 1. Find the center of mass of the solid of constant density 1 bounded between the paraboloid $z = x^2 + y^2$ and the plane $z = 4$. We consider the region to have uniform density 1.

Exercise 2. Convert the following integral to cylindrical coordinates, and then evaluate the result.

$$\int_{y=-1}^{y=1} \int_{x=0}^{x=\sqrt{1-y^2}} \int_{z=0}^{z=x} (x^2 + y^2) dz dx dy.$$

Exercise 3. Write an expression for the volume of the following region D using an integral in cylindrical coordinates. Sketch the region of integration. (You do not need to evaluate the integral). D is the cylinder whose base is contained in the plane $z = 0$, where the base is the region between the circles $r = \cos\theta$ and $r = 2\cos\theta$. Also, the top of D is bounded by the plane $z = 3 - y$.

Exercise 4. Using spherical coordinates, find the volume of the region bounded by the sphere $x^2 + y^2 + z^2 = 1$ and by the cone $z^2 = x^2 + y^2$, where $z \geq 0$.

Exercise 5. Find the moment of inertia about the z-axis, of the region where $x^2+y^2+z^2 \leq 1$ and where $z \leq 0$. We consider the region to have uniform density 1.

Exercise 6. Find the volume of the donut defined in spherical coordinates by $\rho \leq 2\sin\phi$.

Exercise 7. Suppose I want to design a structure of bounded height and with minimal moment of inertia. Specifically, suppose I have a region D in Euclidean space $\mathbf{R}^3$, and D lies between the planes $\{(x, y, z) \in \mathbf{R}^3 \colon z = 0\}$ and $\{(x, y, z) \in \mathbf{R}^3 \colon z = 1\}$. Suppose also that D has uniform density 1, and the mass of D is equal to 1. I then want to find the D with the smallest moment of inertia around the z axis. Which D should I use?

Exercise 8. Suppose $f(x, y)$ is a function of two variables such that there exist two single-variable functions g, h with $f(x, y) = g(x)h(y)$. Show that

$$\int_{[a,b]\times[c,d]} f dA = (\int_a^b g(x)dx)(\int_c^d h(y)dy).$$

Exercise 9. Let X and Y be random variables with joint probability density function

$$p(x, y) = \begin{cases} \frac{1}{72}(2xy + 2x + y) & \text{, if } 0 \leq x \leq 4 \text{ and } 0 \leq y \leq 2 \\ 0 & \text{, otherwise} \end{cases}.$$

- Calculate the probability $P(0 \leq X \leq 2; 1 \leq Y \leq 2)$.

- Calculate the probability that $X + Y \leq 2$.

Exercise 10. Using the spherical coordinate ρ, define the function

$$\psi(\rho) = \frac{1}{\sqrt{\pi a^3}} e^{-\rho/a}.$$

Here ψ is known as the wave function for the $1s$ state of an electron in a hydrogen atom, and $a \approx 5.3 \times 10^{-11}$ is known as the Bohr radius. You may have referred to this function as an s-orbital (or 1s-orbital) in chemistry class. In chemistry class, you also drew a picture of the s-orbital. This picture is sensible but still deceiving, as we will see in this exercise.

Let $p \colon \mathbf{R}^3 \to \mathbf{R}$ be the function defined in spherical coordinates by

$$p(\rho) = |\psi(\rho)|^2 .$$

Then p represents the probability of finding the electron in a certain region of space as follows. Let D be a region in Euclidean space $\mathbf{R}^3$. The probability of finding the electron in the region D is equal to

$$\iiint_D p\, dV.$$

Computing the probability of finding a particle in a certain region of space is a foundational concept in quantum mechanics.

Using integration in spherical coordinates, verify that $\iiint_{\mathbf{R}^3} p\, dV = 1$, so p actually represents a probability. Then, show that the probability of finding an electron at a distance greater than a from the origin is equal to $5/e^2 \approx .677$. Since this probability is around $1/2$, and since this probability goes quickly to 0 when the distance in question becomes larger than the Bohr radius a, we draw the $1s$ orbital as a ball centered at the origin. However, as we can see by the definition of p, there is always a (very small) chance that the electron can be very far from the origin. In this sense, the drawing from chemistry class is misleading.

The other wave functions, which include the p, d and f orbitals that you learned in chemistry, can also be written down as formulas like this, but the formulas become slightly more complicated.