126 Final Solutions[1]

1. Question 1

(a) The function $f(x) = \frac{1}{2}\ln(x+1) + e^x$ is one-to-one (you do not need to verify that fact). Find the equation of the tangent line to $y = f^{-1}(x)$ at the point $x = 1$.

(ii) $y = \frac{2}{3}x - \frac{2}{3}$. Since $f(0) = 1$, $f^{-1}(1) = 0$, so $(d/dx)f^{-1}(1) = 1/[f'(f^{-1}(1))] = 1/f'(0) = 1/(3/2) = 2/3$. So, the tangent line has slope $2/3$, and it intersects the point $(1,0)$.

(b) The length of the function $f(x) = \frac{x^3}{6} + \frac{1}{2x}$, with $x \in [\frac{1}{2}, 1]$ is given by:

(iv) $\frac{1}{2}\int_{1/2}^{1} x^2 + x^{-2}\,dx$. The length is

$$\int_{1/2}^{1}\sqrt{1 + ((1/2)(x^2 - x^{-2}))^2}dx = \int_{1/2}^{1}\sqrt{1 + (1/4)x^4 + (1/4)x^{-4} - 1/2}dx$$
$$= \int_{1/2}^{1}\sqrt{1/2 + (1/4)x^4 + (1/4)x^{-4}}dx = (1/2)\int_{1/2}^{1}\sqrt{2 + x^4 + x^{-4}}dx$$
$$= (1/2)\int_{1/2}^{1}\sqrt{(x^2 + x^{-2})^2}dx = (1/2)\int_{1/2}^{1}(x^2 + x^{-2})dx$$

(c) Consider the following series

$$\text{(I)}\ \sum_{n=3}^{+\infty}\frac{\sin(\frac{1}{n})}{(\frac{1}{n})} \qquad \text{(II)}\ \sum_{n=3}^{+\infty}\frac{(-1)^n}{\sqrt[3]{n-1}}$$

Which one of the following statements is true?

(i) (I) is divergent and (II) is conditionally convergent. Since $\lim_{n\to\infty} n\sin(1/n) = \lim_{x\to 0}\sin(x)/x = 1$, the I diverges by the Divergence Test. By the integral test, II diverges absolutely. By the alternating series test, II converges. That is, II converges conditionally.

(d) $\int_{-1}^{0}\frac{dx}{x^2-4x+3}$ is equal to

(i) $(1/2)\ln(3)+(1/2)\ln(2)-(1/2)\ln(4)$. We can write $1/(x^2-4x+3) = A/(x-1)+B/(x-3)$, i.e. $1 = A(x-3) + B(x-1)$, i.e. $A(-2) = 1$, $B(3-1) = 1$, so $A = -1/2$, $B = 1/2$, and $\int_{-1}^{0} 1/(x^2-4x+3)dx = -(1/2)\int_{-1}^{0} dx/(x-1) + (1/2)\int_{-1}^{0} dx/(x-3) = [(-1/2)\ln|x-1| + (1/2)\ln|x-3|]_{x=-1}^{x=0} = (1/2)\ln(3) + (1/2)\ln(2) - (1/2)\ln(4) = (1/2)\ln(6/4) = (1/2)\ln(3/2)$.

(e) Which of the following is $\int\frac{f'(x)}{g(x)}dx$ equal to?

(v) none of the above. Upon taking derivatives, none of the expressions are equal to the first expression.

(f) Which of the following integrals converges?

(iv) $\int_{2}^{\infty}\frac{x^3}{x^5-1}dx$. The function behaves like x^{-2} near infinity, so it converges by integral comparison. ($\frac{x^3}{x^5-1} \leq 2x^{-2}$, since $x^5 \leq 2[x^5-1]$, since $0 \leq x^5 - 2$ for all $x \geq 2$, so $0 \leq \int_{2}^{\infty}\frac{x^3}{x^5-1}dx \leq \int_{2}^{\infty} 2x^{-2}dx < \infty$.)

2. Question 2

Consider the surface S obtained by rotating the curve $y = e^{-x}$ about the x–axis, with $x \in [0, +\infty)$.

[1] December 12, 2019,

(a) Establish that the surface area of S is given by $A = 2\pi \int_0^1 \sqrt{1+u^2}\, du$

Solution. The surface area is given by

$$\int_0^\infty 2\pi e^{-x}\sqrt{1+((d/dx)e^{-x})^2}dx = \int_0^\infty 2\pi e^{-x}\sqrt{1+e^{-2x}}dx = -\int_1^0 2\pi\sqrt{1+u^2}du = 2\pi\int_0^1 \sqrt{1+u^2}du$$

Here we used the substitution $u = e^{-x}$, so that $du = -e^{-x}dx$. Also the last equality used $\int_a^b f(x)dx = -\int_b^a f(x)dx$.

(b) Calculate the value of the integral A.

Solution. We use the trig substitution $u = \tan\theta$, so that $du = \sec^2\theta$ and

$$\begin{aligned}\int_0^1 \sqrt{1+u^2}du &= \int_0^{\pi/4} \sqrt{1+\tan^2\theta}\sec^2\theta d\theta = \int_0^{\pi/4} |\sec\theta|\sec^2\theta d\theta = \int_0^{\pi/4} \sec^3\theta d\theta \\ &= [(1/2)\sec\theta\tan\theta]_{\theta=0}^{\theta=\pi/4} + (1/2)\int_0^{\pi/4}\sec\theta d\theta \\ &= [(1/2)\sec\theta\tan\theta]_{\theta=0}^{\theta=\pi/4} + (1/2)\ln(\sec\theta+\tan\theta)_{\theta=0}^{\theta=\pi/4} \\ &= (1/2)(\sqrt{2}+\ln(\sqrt{2}+1)).\end{aligned}$$

Note that $\sec\theta \geq 0$ when $0 \leq \theta \leq \pi/4$, so we were justified in removing the absolute values.

3. Question 3

Consider a torus (donut) of major radius R and minor radius r $(0 < r < R)$.

(a) Show that the volume of the torus is given by the integral

$$V = 4\pi R\int_{-r}^r \sqrt{r^2-x^2}\, dx - 4\pi\int_{-r}^r x\sqrt{r^2-x^2}\, dx.$$

Solution. Note that $x^2+y^2=r^2$, so $y^2 = r^2-x^2$, so (if we take the positive square root), $y = \sqrt{r^2-x^2}$. Since rotating this curve will only give the volume of the "top half" of the donut, we use $f(x) = 2\sqrt{r^2-x^2}$ to also get the "bottom half" of the donut. Using volumes by revolution (cylindrical shells) we have, with $f(x) = 2\sqrt{r^2-x^2}$,

$$\begin{aligned}V &= 2\pi\int_{x=-r}^{x=r}(R-x)f(x)dx = 2\pi\int_{x=-r}^{x=r}(R-x)2\sqrt{r^2-x^2}dx \\ &= 4\pi R\int_{x=-r}^{x=r}\sqrt{r^2-x^2}dx - \int_{x=-r}^{x=r}\sqrt{r^2-x^2}dx.\end{aligned}$$

(b) Calculate the value of $J = \int_{-r}^r x\sqrt{r^2-x^2}\, dx$.

Solution. The function is odd, and it is integrated on $[-r, r]$, so this integral is zero. Alternatively, using the substitution $u = r^2-x^2$, with $du = -2xdx$, we have

$$\int_{-r}^r x\sqrt{r^2-x^2}\, dx = \int_0^0 u^{1/2}(-(1/2))du = 0.$$

(c) Calculate the value of $I = \int_{-r}^r \sqrt{r^2-x^2}\, dx$. Deduce the value of V.

Solution. Using the trigonometric substitution $x = r\sin\theta$, we have $dx = r\cos\theta d\theta$, and

$$I = \int_{-\pi/2}^{\pi/2} r\sqrt{1-\sin^2\theta}\, r\cos\theta d\theta = \int_{-\pi/2}^{\pi/2} r\,|\cos\theta|\, r\cos\theta d\theta$$

$$= r^2\int_{-\pi/2}^{\pi/2}\cos^2\theta d\theta = r^2\int_{-\pi/2}^{\pi/2}(1/2)(1+\cos 2\theta)d\theta = r^2(1/2)[\theta+(1/2)\sin 2\theta]_{-\pi/2}^{\pi/2} = r^2(1/2)(\pi).$$

Note that $\cos\theta \geq 0$ when $-\pi/2 \leq \theta \leq \pi/2$, so the removal of absolute values is justified.

We conclude that

$$V = 4\pi RI = 2\pi^2 Rr^2.$$

4. Question 4

Consider $f(x) = \frac{2}{3-x}$ on $[0,2]$. Note that, range$(f) \subseteq [0,2]$. (You do not need to show this fact and may use it freely.) Let $(u_n)_n$ be the sequence defined by $u_0 = \frac{3}{2}$ and $u_{n+1} = f(u_n)$, $\forall n \geq 0$.

(a) Show that, for any $n \geq 0$, $u_n \in [0,2]$. You must use a mathematical induction to prove this assertion.

Solution. The base case holds since $u_0 \in [0,2]$. We now induct on n. We assume $u_n \in [0,2]$ and we need to show that $u_{n+1} \in [0,2]$. Note that $f'(x) = 2/(3-x)^2 > 0$, so that f is an increasing function on $[0,2]$. That is, for any $x \in [0,2]$, $f(0) \leq f(x) \leq f(2)$. That is, for any $x \in [0,2]$, $2/3 \leq f(x) \leq 1$. By the inductive hypothesis, $u_n \in [0,2]$, so that $u_{n+1} = f(u_n) \in [2/3,1] \subseteq [0,2]$. Having completed the inductive step, the assertion follows by induction.

(b) Show that $(u_n)_n$ is decreasing. You must use a mathematical induction to prove this assertion. **Hint:** Use the fact that f is increasing.

Solution. The base case holds since $u_0 = 3/2$ and $u_1 = 2/(3/2) = 4/3$ so that $u_1 < u_0$. We now induct on n. We assume $u_{n+1} < u_n$ and we need to show that $u_{n+2} < u_{n+1}$. Since $0 \leq u_{n+1} < u_n \leq 2$ by assumption (and part (a)), as noted in part (a) f is increasing on $[0,2]$, so that $f(u_{n+1}) < f(u_n)$, i.e. $u_{n+2} < u_{n+1}$. Having completed the inductive step, the assertion follows by induction.

(c) Show that $(u_n)_n$ converges to a limit ℓ, then determine the value of ℓ. You must carefully justify your answer.

Solution. The sequence $u_0, u_1, \ldots$ is monotonic and bounded by parts (b) and (a), respectively. It therefore converges. Let $L = \lim_{n\to\infty} u_n$. Then, using the definition of u_{n+1} and the continuity of f on $[0,2]$,

$$L = \lim_{n\to\infty} u_n = \lim_{n\to\infty} u_{n+1} = \lim_{n\to\infty} f(u_n) = f(\lim_{n\to\infty} u_n) = f(L).$$

That is, $L = f(L)$. That is, $L = 2/(3-L)$, i.e. $-L^2 + 3L - 2 = 0$, i.e. $L = [-3 \pm \sqrt{9-8}]/(-2) = (3/2) \pm 1/2$. That is, $L = 1$ or $L = 2$. Since $u_0 = 3/2$ and the sequence is decreasing, we conclude that $L = 1$.

5. Question 5

(b) Determine whether or not the following series converges: $\sum_{n=1}^{\infty} \frac{3^{(n^2)}}{n!}$.

Solution. The series diverges. Let $a_n = 3^{n^2}/n!$. From the ratio test, we have $|a_{n+1}/a_n| = 3^{2n+1}/(n+1) \to \infty$ as $n \to \infty$. So, $\lim_{n\to\infty} |a_{n+1}/a_n| = \infty$, so the series diverges from the ratio test.

(b) Determine whether the following series converges absolutely, converges conditionally, or diverges.

$$\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}.$$

Solution. This series converges absolutely. That is, the series $\sum_{n=1}^{\infty} |\cos(n)|/n^2$ converges. This follows from the comparison test. Note that $|\cos(n)| \leq 1$, so the series $\sum_{n=1}^{\infty} |\cos(n)|/n^2$ converges because the series $\sum_{n=1}^{\infty} 1/n^2$ converges.

(c) Let $0 < r < s$ be constants. Does the sequence $\{(r^n + s^n)^{1/n}\}$ converge as $n \to \infty$? If it does converge, what is the limit?

Solution. This sequence converges to s. To see this, note that since $r < s$, we have $s^n \leq r^n + s^n \leq s^n + s^n = 2s^n$. So, $s \leq (r^n + s^n)^{1/n} \leq 2^{1/n}s$. Letting $n \to \infty$, $2^{1/n} \to 1$. So, $s \leq \lim_{n\to\infty}(r^n + s^n)^{1/n} \leq s$. By the squeeze theorem, $\lim_{n\to\infty}(r^n + s^n)^{1/n} = s$.

6. Question 6

(a) Suppose a particle is moving back and forth along the x-axis. The particle begins at the point $x = 0$. It then moves in the positive x-direction a distance of 1. The particle then moves in the negative x-direction a distance of $1/4$. The particle then moves in the positive x-direction a distance of $1/16$. The particle then moves in the negative x-direction a distance of $1/64$, and so on. In general, whenever the particle moves a distance h in one direction, it then moves in the opposite direction a distance of $h/4$. What position on the x-axis does the particle eventually approach?

Solution. We need to compute $x = 1 - 1/4 + (1/4)^2 - \cdots = \sum_{n=0}^{\infty}(-1/4)^n = 1/(1 - (-1/4)) = 1/(5/4) = 4/5$. That is, we just summed a geometric series.

(b) Evaluate

$$\sum_{n=0}^{\infty} \int_n^{n+1} xe^{-x}dx.$$

Solution. $\sum_{n=0}^{\infty} \int_n^{n+1} xe^{-x}dx = \int_0^{\infty} xe^{-x}dx$. Integrating by parts,

$$\int_0^{\infty} xe^{-x}dx = \lim_{N\to\infty} \int_0^N x(d/dx)(-e^{-x})dx = \lim_{N\to\infty} [-xe^{-x}]_0^N + \int_0^N e^{-x}dx$$
$$= \lim_{N\to\infty} -Ne^{-N} + (1 - e^{-N}) = 1.$$

7. Question 7

For the following power series, find the radius of convergence R. Describe the set of all points where the power series converges absolutely, describe the set of all points where the power series converges conditionally, and describe the set of all points where the power series diverges.

$$\sum_{n=1}^{\infty}(n+1)x^{n-1}.$$

Solution. Let x be a real number and let $a_n = (n+1)x^{n-1}$. Note that $|a_{n+1}/a_n| = \frac{n+2}{n+1}|x| \to |x|$ as $n \to \infty$. So, from the ratio test, the power series converges absolutely when $|x| < 1$ and it diverges when $|x| > 1$. That is, the radius of convergence is $R = 1$. It remains to check what happens when $|x| = 1$. When $x = 1$, we have the series $\sum_{n=0}^{\infty}(n+1)$, which diverges by the Divergence Test. Also, when $x = -1$, we have the series $\sum_{n=0}^{\infty}(n+1)(-1)^{n-1}$ which again diverges by the Divergence Test. So, the series never converges conditionally, it diverges when $|x| \geq 1$ and it converges absolutely when $|x| < 1$.

Finally, note that $1/(1-x) = \sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} x^{n-1}$, and differentiating both sides gives $1/(1-x)^2 = \sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^{n-1}$ (since the first term in the sum is zero). So,

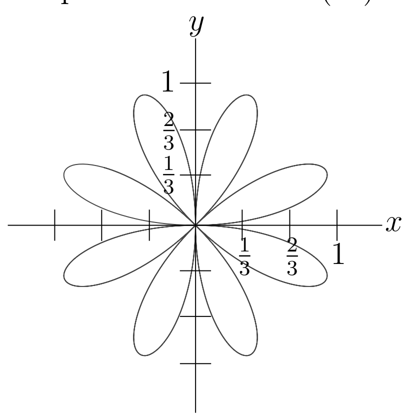
$$\sum_{n=1}^{\infty}(n+1)x^{n-1} = \sum_{n=1}^{\infty} x^{n-1} + \sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{1-x} + \frac{1}{(1-x)^2} = \frac{2-x}{(1-x)^2}.$$

8. Question 8

Consider the equation

$$r = \sin(4\theta).$$

(a) In the figure below, draw the polar curve $r = \sin(4\theta)$ for all $0 \leq \theta \leq 2\pi$.



(b) Find the area enclosed by the curve, in the first quadrant. (The first quadrant is the region where $x \geq 0$ and $y \geq 0$).

Solution. One petal is formed when $0 \leq \theta \leq \pi/4$, so the area of one petal is

$$\int_0^{\pi/4} \frac{1}{2}(r(\theta))^2 d\theta = \int_0^{\pi/4} \frac{1}{2}\sin^2 4\theta d\theta = \int_0^{\pi/4} (1/4)(1-\cos 8\theta)d\theta$$
$$= (1/4)[\theta - (1/8)\sin 8\theta]_{\theta=0}^{\theta=\pi/4} = \pi/16.$$

So, the area formed by two petals is $\pi/8$. This is the area of the region in the first quadrant.

(c) Find the slope of the tangent when $\theta = \pi/4$.

Solution. The curve $s(\theta) = (r(\theta)\cos\theta, r(\theta)\sin\theta)$ has tangent vector $s'(\theta) = r(\theta)(-\sin\theta, \cos\theta) + r'(\theta)(\cos\theta, \sin\theta)$. So,

$$s'(\pi/4) = 0 + (-4)(\sqrt{2}/2, \sqrt{2}/2) = (-2\sqrt{2}, -2\sqrt{2}).$$

So, the slope is rise (change in y) over run (change in x), i.e.

$$\frac{-2\sqrt{2}}{-2\sqrt{2}} = 1.$$

(d) Write an integral that computes the arc length of this curve where $0 \le \theta \le \pi/4$. You do NOT have to evaluate this integral.

$$\int_a^b \sqrt{r(\theta)^2 + r'(\theta))^2} d\theta = \int_0^{\pi/4} \sqrt{\sin^2(4\theta) + 16\cos^2(4\theta)} d\theta.$$