126 Midterm 1 Solutions[1]

1. Question 1

True/False

(a) Let x be a real number. Then $\cos^{-1}(\cos(x)) = x$.

FALSE. Let $x = 3\pi/2$. Then $\cos(3\pi/2) = 0$, but $\cos^{-1}(0) = \pi/2$, so $\cos^{-1}(\cos(3\pi/2)) = \pi/2 \neq 3\pi/2$.

(b) The function $f(x) = x^2$ with domain $[-1, 1]$ is invertible.

FALSE. f fails the horizontal line test. That is, $f(1) = f(-1) = 1$, so f is not invertible with domain $[-1, 1]$.

(c) Let f be a function of a real variable with inverse g. If x is in the domain of g and if $f'(g(x)) \neq 0$, then $g'(x) = \dfrac{1}{f'(g(x))}$.

TRUE. Recall that $f(g(x)) = x$, so by the chain rule, $1 = f'(g(x))g'(x)$, so $g'(x) = 1/f'(g(x))$.

(d) Let $f(x) = e^x$ with domain $(-\infty, \infty)$. Then the inverse of f is the function $\ln(x)$ with domain $(0, \infty)$.

TRUE.

(e) Let $\sin^{-1}(x)$ denote the inverse sine of x. Let $-1 < x < 1$. Then $\cos(\sin^{-1}(x)) = \sqrt{1 - x^2}$.

TRUE. If $\theta = \sin^{-1}(x)$, then $\sin\theta = x$, i.e. θ is the angle of a right triangle with hypotenuse 1 and side lengths x and $\sqrt{1 - x^2}$. So, $\cos\sin^{-1} x = \cos\theta = \sqrt{1 - x^2}/1$.

2. Question 2

Compute the following limits.

(a) $\displaystyle\lim_{x\to 0^+} \frac{\cos(x) - 1}{x^2}$.

Solution. Using L'Hopital's Rule twice, we have

$$\lim_{x\to 0^+} (\cos(x) - 1)/x^2 \overset{L'H}{=} \lim_{x\to 0^+} -\sin(x)/2x \overset{L'H}{=} \lim_{x\to 0^+} (-\cos x)/2 = -1/2.$$

(b) $\displaystyle\lim_{x\to 0} \frac{x+2}{x+1}$.

Solution. Since the function $f(x) = (x + 2)/(x + 1)$ is continuous at $x = 0$, we can simply substitute $x = 0$ to get $\lim_{x\to 0}(x + 2)/(x + 1) = 2/1 = 2$.

(c) $\displaystyle\lim_{x\to\infty} \tan^{-1}(x)$.

Solution. Since $\lim_{x\to\pi/2^-} \tan(x) = +\infty$, we have $\lim_{x\to\infty} \tan^{-1}(x) = \pi/2$.

3. Question 3

Evaluate the following integrals.

(a) $\int x \ln x dx$.

Solution. Integrating by parts, we have $\int x \ln x = \int(\ln x)(d/dx)(x^2/2) = (x^2/2)\ln x - \int(x^2/2)(1/x) = (x^2/2)\ln x - (1/2)\int x = (x^2/2)\ln x - (x^2/4) + C$.

(b) $\int_4^5 t^{-1}(\ln t)^{-1} dt$.

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Solution. Substituting $u = \ln t$, so that $du = (1/t)dt$, we have $\int_4^5 t^{-1}(\ln t)^{-1}dt = \int_{\ln 4}^{\ln 5} u^{-1}du = \ln\ln 5 - \ln\ln 4$.

4. Question 4

(a) $\int_0^{\pi/2} \cos^5(x)dx$.

Solution. $\int_0^{\pi/2} \cos^5(x)dx = \int_0^{\pi/2}(1-\sin^2(x))^2\cos(x)dx = \int_0^1(1-u^2)^2du = \int_0^1(1-2u^2+u^4)du = 1-(2/3)+(1/5) = 8/15$.

(b) $\int \frac{t}{t^2-3t+2}\,dt$

Solution. We solve for A, B in

$$\frac{t}{t^2-3t+2} = \frac{t}{(t-2)(t-1)} = \frac{A}{t-2} + \frac{B}{t-1}.$$

That is, we solve $t = A(t-1) + B(t-2)$. When $t = 1$, we have $B = 1/(-1) = -1$. Then $t = 2$, we have $A(1) = 2$. So,

$$\frac{t}{t^2-3t+2} = \frac{2}{t-2} + \frac{-1}{t-1}.$$

Therefore, $\int \frac{t}{t^2-3t+2}dt = 2\ln|t-2| - \ln|t-1| = \ln\left(\frac{(t-2)^2}{|t-1|}\right) + C$.

5. Question 5

$\int_{\sqrt{2/3}}^{\sqrt{2}} x^{-4}\sqrt{2+x^2}\,dx$.

Solution. Let $x = \sqrt{2}\tan\theta$, so $dx = \sqrt{2}(\cos\theta)^{-2}d\theta$, and $\sqrt{2+x^2} = \sqrt{2}\sqrt{1+\tan^2\theta} = \sqrt{2}\,|\cos\theta|^{-1} = \sqrt{2}(\cos\theta)^{-1}$ (we can get rid of the absolute values since $\sqrt{2/3} \leq x \leq \sqrt{2}$ implies that $\pi/6 \leq \theta \leq \pi/4$, and $\cos\theta$ is positive in this region). So, $\int_{\sqrt{2/3}}^{\sqrt{2}} x^{-4}\sqrt{2+x^2}dx = \int_{\pi/6}^{\pi/4}(1/2)(\tan\theta)^{-4}(\cos\theta)^{-3}d\theta = (1/2)\int_{\pi/6}^{\pi/4}\cos\theta(\sin\theta)^{-4}d\theta = (1/2)\int_{1/2}^{\sqrt{2}/2} u^{-4}du = (1/2)[-(1/3)u^{-3}]_{u=1/2}^{u=1/\sqrt{2}}$ $(1/6)(8-2^{3/2}) = (4-\sqrt{2})/3$. Here we used the substitution $u = \sin\theta$ so that $du = \cos\theta$ and $1/2 \leq u \leq \sqrt{2}/2$.