Digest 9

(A compilation of emailed homework questions, answered around Wednesday.)

Question. (From a student): Does the divergence test only apply to series that begin at n=1 ?

Answer. No. For example, let $\{a_n\}$ be a sequence and suppose we start at $n = 0$ instead of $n = 1$. Then $\sum_{n=0}^{\infty} a_n$ diverges if and only if $\sum_{n=1}^{\infty} a_n$. And Then $\sum_{n=0}^{\infty} a_n$ diverges if and only if $\sum_{n=53}^{\infty} a_n$ diverges. And so on. To be even more specific, consider $a_n = n$. Then $\sum_{n=0}^{\infty} a_n = 0 + 1 + 2 + 3 + \cdots$ diverges, and so does $\sum_{n=20}^{\infty} a_n = 20 + 21 + 22 + 23 + \cdots$.

As far as convergence and divergence are concerned, the place where we start the sum does not matter.

Question. (From a student): This question refers to number 9 from 11.3 in the book. It asks to use the integral test for the infinite series:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

So I took its integral and I solved that it would be

$$\lim_{R\to\infty} \int_1^R \frac{1}{x(x+1)} dx = \ln(R)\ln(R+1) - \ln(1)\ln(2).$$

That equals infinity and diverges yes? Perhaps I'm doing it wrong, the book say that it converges.

Answer. The integration is incorrect. You do want to compute the integral of $\frac{1}{x(x+1)}$. However, you might want to use something like $\frac{1}{x(x+1)} \leq \frac{1}{x^2}$ when $x \geq 1$. Then

$$\lim_{R\to\infty} \int_1^R \frac{1}{x(x+1)} dx \leq \lim_{R\to\infty} \int_1^R \frac{1}{x^2} dx = \lim_{R\to\infty} [-x^{-1}]_1^R = \lim_{R\to\infty} [-R^{-1} + 1] = 1.$$

So, we see that the integral converges. So the sum converges.

Question. [From a student] In Example 1 of 11.7 (pg. 598), it says: Find the Taylor series for $f(x) = x^{-3}$, where $c = 1$. So, $f'(x) = -3x^{-4}$, and $f''(x) = 12x^{-5}$, and in general $f^{(n)}(x) = (-1)^n(3)(4)\cdots(n+2)x^{-3-n}$

Why is the last term multiplied by $(n+2)$?

Answer. We know that $f'(x) = -3x^{-4}$ and $f''(x) = 12x^{-5}$. In general, we claim that

$$f^{(n)}(x) = (-1)^n 3 \cdot 4 \cdot 5 \cdots n(n+1)(n+2)x^{-3-n}. \qquad (*)$$

We can prove this formula holds by induction on n. The case $n = 1$ holds, since $f'(x) = f^{(1)}(x) = (-1)^1 3x^{-3-1} = -3x^{-4}$. (Note that $n + 2 = 3$ when $n = 1$.) So, to prove that the equation $(*)$ holds, we consider the inductive step. Suppose $(*)$ holds for n. We must then prove that it holds for the case $n+1$. To see this, we use that $(*)$ holds for the case n to get

$$\begin{aligned} f^{(n+1)}(x) = \frac{d}{dx} f^{(n)}(x) &= (-1)^n 3 \cdot 4 \cdot 5 \cdots n(n+1)(n+2) \frac{d}{dx} x^{-3-n} \\ &= (-1)^n 3 \cdot 4 \cdot 5 \cdots n(n+1)(n+2)(-3-n)x^{-3-n-1} \\ &= (-1)^n (-1) 3 \cdot 4 \cdot 5 \cdots n(n+1)(n+2)(n+3)x^{-3-(n+1)} \\ &= (-1)^{n+1} 3 \cdot 4 \cdot 5 \cdots (n+1)((n+1)+1)((n+1)+2)x^{-3-(n+1)}. \end{aligned}$$

So, formula $(*)$ holds for $n+1$. So, it holds for all n. (I will not expect you guys to do proofs by induction in this class; this method can just help you understand how to find certain Taylor series.)

Finally, formula $(*)$ tells us the Taylor series for f, since $(*)$ implies that

$$f^{(n)}(1) = (-1)^n 3 \cdot 4 \cdot 5 \cdots n(n+1)(n+2).$$

So, the Taylor series is

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n &= \sum_{n=0}^{\infty} \frac{(-1)^n 3 \cdot 4 \cdot 5 \cdots n(n+1)(n+2)}{1 \cdot 2 \cdot 3 \cdots (n-1)n}(x-1)^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(n+2)}{2}(x-1)^n \end{aligned}$$