31B: CALCULUS 2, INTEGRATION AND INFINITE SERIES, SPRING 2015

STEVEN HEILMAN

Contents

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1. Introduction

As you have covered in your previous calculus class, the subject of calculus has many applications. For example, calculus is very closely related to **probability**, which itself has applications to **statistics** and **algorithms**. For example, the first generation of Google's search technology came from ideas from probability theory. Within **physics**, differential equations often arise. In **economics**, optimization is often used to e.g. maximize profit margins. Also, the ideas of single variable calculus are developed and generalized within financial mathematics to e.g. **stochastic calculus**. Biology uses many ideas from calculus. Signal processing and **Fourier analysis** provide some nice applications within many areas of science. For example, our cell phones use Fourier analysis to compress voice signals.

In this course, we will begin with the exponential function, one of the most important functions in mathematics, due to its appearance in many applications. We will then discuss various methods of integration that were not covered in your previous course. We will then conclude the course with a thorough treatment of Taylor series, which is a way of writing a function as an infinite sum of monomials. Taylor series is quite useful, since it allows a nice understanding of potentially complicated functions.

2. The Exponential and the Logarithm

2.1. Exponential Functions.

From our previous courses (or our calculators), we should have a good understanding of the meaning of the following numbers

$$2^1 = 2, \qquad 3^2 = 9, \qquad 6^{-1} = 1/6, \qquad (1/2)^{1.5} = 2^{-3/2} \approx 0.354.$$

Perhaps it is less obvious what is meant by the numbers

$$2^\pi, \qquad 0^0, \qquad (-1)^{\sqrt{2}}$$

We will eventually understand the first two numbers, but the last one will not be discussed in this class (though calculators seem to have an idea of what the last number should be). We would like to give a general treatment of numbers of this sort, since they arise so frequently. We therefore will discuss exponential functions below. There are several ways to introduce exponential functions, and we will try to describe these different presentations later on, since they are all useful.

Definition 2.1. Let $b > 0$ with $b \neq 1$. Let x be a real number. An **exponential function** is a function of the form

$$f(x) = b^x.$$

(Since $1^x = 1$ for all real numbers x, the case $b = 1$ is excluded.)

Here are some properties of exponential functions.

Proposition 2.2 (**Properties of Exponential Functions**). *Let $b > 0$, let x, y be real numbers, let n be a positive integer, and let $f(x) = b^x$.*

- *Exponential functions are always positive: $b^x > 0$ for all real numbers x.*
- *If $b > 0$ with $b \neq 1$, the range of $f(x) = b^x$ is the set of all positive real numbers.*
- *If $b > 1$, then $f(x) = b^x$ is increasing. If $0 < b < 1$, then f is decreasing.*
- *$b^0 = 1$.*
- *$b^{x+y} = b^x b^y$.*

- $\frac{b^x}{b^y} = b^{x-y}$.
- $b^{-x} = \frac{1}{b^x}$.
- $(b^x)^y = b^{(xy)}$.
- $b^{1/n} = \sqrt[n]{b}$.

2.1.1. *Derivatives of Exponential Functions.* Let $b > 0$, let x be a real number, and let $f(x) = b^x$. Let's try to compute the derivative of $f(x)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{b^{x+h} - b^x}{h} = \lim_{h\to 0} \frac{b^x b^h - b^x}{h}$$
$$= \lim_{h\to 0} b^x \frac{b^h - 1}{h} = b^x \lim_{h\to 0} \frac{b^h - 1}{h} = b^x m(b).$$

Here we defined $m(b) = \lim_{h\to 0} \frac{b^h-1}{h}$. In summary,

Proposition 2.3. *Let $b > 0$. The function $f(x) = b^x$ is proportional to its own derivative. That is,*

$$f'(x) = m(b)f(x).$$

We will compute $m(b)$ later on, but for now, let's find a case where $m(b) = 1$.

Note that $m(1) = 0$, $m(3)$ seems larger than 1, and m seems to be continuous. So, by the Intermediate Value Theorem, there seems to be a unique number $0 < e < 3$ such that $m(e) = 1$. In summary,

Proposition 2.4. *There is a unique positive real number e such that*

$$\frac{d}{dx} e^x = e^x.$$

Also, $e \approx 2.718281828\ldots$.

Remark 2.5. Let $b > 0$, $f(x) = b^x$. Then $f(0) = 0$, and $f'(0) = m(b)$ is the slope of f at $x = 0$.

Remark 2.6. The number e was actually discovered in an investigation related to compound interest. We will discuss this later on, but we note for now that

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

And more generally,

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n.$$

The last formula is natural in the following sense. Suppose I have 100 dollars in a bank that computes interest compounded once a year, at a 6 percent interest rate. After a year, I then have

$$100(1 + .06).$$

If instead, the bank compounds the interest daily at the 6 percent rate, then after a year I have

$$100(1 + .06/(365))^{365}$$

If instead, the bank compounds the interest every second at the 6 percent rate, then after a year I have

$$100(1 + .06/(31536000))^{31536000}$$

So, for $x = .06$, e^x computes the money that I have after a year when the interest rate is x, and when the compounding time goes to zero.

Proposition 2.7. *Let $g\colon \mathbb{R} \to \mathbb{R}$ be a function. Then*

$$\frac{d}{dx}e^{g(x)} = g'(x)e^{g(x)}.$$

In particular, for c, d constants, we have $(d/dx)e^{cx+d} = ce^{cx+d}$.

Proposition 2.8. *Let $a < b$ and let c, d be real numbers with $c \neq 0$. Then*

$$\int e^x dx = e^x + C, \qquad \int_a^b e^x dx = \int_a^b (d/dx)e^x dx = e^b - e^a.$$

$$\int e^{cx+d} dx = c^{-1}e^{cx+d} + C, \qquad \int_a^b e^{cx+d} dx = \int_a^b (d/dx)c^{-1}e^{cx+d} dx = c^{-1}(e^{bc+d} - e^{ac+d}).$$

2.2. **Inverse Functions.** Let f be a real valued function on the real line. An inverse function for f does not always exist. If an inverse function for f exists, it will un-do the effect of f. For example, the function $f(x) = x^3$ has an inverse $f^{-1}(x) = x^{1/3}$. Note that

$$f(f^{-1}(x)) = (x^{1/3})^3 = x, \qquad f^{-1}(f(x)) = (x^3)^{1/3} = x.$$

The function $f(x) = x^3$ is a bit special, in that it actually has an inverse function. To see what makes this function special, let's consider an example that does not have an inverse. Consider the function $f(x) = x^2$ on the whole real line. Note that $f(2) = f(-2) = 4$. This implies that f does not have an inverse. If we could find a function f^{-1} that un-does the effect of f, then when we apply f^{-1} to the equality $f(2) = f(-2)$, we would get $2 = -2$, which is clearly false. Let's therefore take this obstruction to creating an inverse, and turn it into a definition.

Definition 2.9. Let f be a function with domain D and range R. We say that f is **one-to-one** if and only if, for every $c \in R$, there exists exactly one $x \in D$ such that $f(x) = c$.

Definition 2.10. Let f be a function with domain D and range R. If there is a function g with domain R and range D such that

$$g(f(x)) = x \quad \text{for all}\, x \in D \qquad \text{and} \qquad f(g(x)) = x \quad \text{for all}\, x \in R,$$

then f is said to be **invertible**. The function g is called the **inverse** of f, and we denote it by $g = f^{-1}$.

Remark 2.11. If f is a one-to-one function with domain D and range R, then f is invertible. If f is not one-to-one, then f is not invertible.

Example 2.12. The function $f(x) = x^3$ with domain $(-\infty, \infty)$ and range $(-\infty, \infty)$ is one-to-one, since any real number c has exactly one cube root x with $x^3 = c$. So, f is invertible. As we showed above, $f^{-1}(x) = x^{1/3}$.

Example 2.13. The function $f(x) = x^2$ with domain $(-\infty, \infty)$ and range $[0, \infty)$ is not one-to-one, since, as we saw above, the number 4 has two distinct numbers 2 and -2 such that $f(2) = f(-2) = 4$. So, f is not invertible when it has the domain $(-\infty, \infty)$. However, if we restrict the domain of f, so that we consider $f(x) = x^2$ with domain $[0, \infty)$ and range $[0, \infty)$, then f is one-to-one. Every nonnegative number c has exactly one nonnegative square root

x such that $x^2 = c$. So, f is invertible when it has the domain $[0, \infty)$. In this case, f^{-1} is the square root function, which we write as $f^{-1}(x) = \sqrt{x}$. (Note that the square root function $\sqrt{x}$ is distinct from the concept of "the square roots of x." In particular, $\sqrt{x}$ is only defined when $x \geq 0$, and it holds that $\sqrt{x^2} = |x|$ for any real number x. So, if x is negative, $\sqrt{x^2} \neq x$. Moreover, the answer to "What are the square roots of 9" is 3 and -3.)

Remark 2.14 (Horizontal Line Test). Let f be a function with domain D and range R. Suppose we look at the graph of f over the domain D. Then f is one-to-one if and only if every horizontal line passes through the graph of f in at most one point.

Remark 2.15. The function $f(x) = x^3$ with domain $(-\infty, \infty)$ satisfies the horizontal line test. The function $f(x) = x^2$ with domain $(-\infty, \infty)$ does not satisfy the horizontal line test.

Proposition 2.16. *Let f be a strictly increasing function. That is, if $x < y$, then $f(x) < f(y)$. Then f is one-to-one.*

Remark 2.17. Recall that a continuously differentiable function with domain $(-\infty, \infty)$ such that $f'(x) > 0$ is increasing. This follows from the Mean Value Theorem. If we had $x < y$ with $f(x) \geq f(y)$, then there would be some c with $x < c < y$ such that $f'(c) = (f(y) - f(x))/(y - x) \leq 0$. But $f'(c) > 0$, so we must have $f(x) < f(y)$, so that f is strictly increasing.

Remark 2.18. Since $(d/dx)e^x = e^x > 0$, the function e^x is increasing on $(-\infty, \infty)$. So, this proposition implies that the exponential function is one-to-one, so that it has an inverse. Similarly, if $b > 1$, then $(d/dx)b^x = m(b)b^x$, and $m(b) > 0$, so b^x is increasing and invertible. And if $b < 1$, then $(d/dx)b^x = m(b)b^x$ and $m(b) < 0$, so $-b^x$ is increasing and invertible.

To better understand the inverse of the exponential function, let's consider the graph of the inverse of the exponential function.

Remark 2.19. Suppose $y = f(x)$ is an invertible function. Then $(x, y) = (x, f(x))$ is a point in the graph of f. Since f is invertible, we can apply f^{-1} to both sides of $y = f(x)$ to get $f^{-1}(y) = x$. So, the point $(y, x) = (y, f^{-1}(y))$ is in the graph of f^{-1}. So, whenever the graph of f contains (x, y), the graph of f^{-1} contains (y, x). Note that (y, x) is the reflection of the point (x, y) across the line $y = x$. Graphically, this means that if we plot the function $y = f(x)$, then the graph of the inverse function f^{-1} is the reflection of the graph of f across the line $y = x$.

Example 2.20. Let's plot the function $f(x) = e^x$. Note that $f(0) = 1$, and $e^x > x$ for all x. This follows since $e^0 > 0$, and since $f'(x) = e^x > 1$ for all $x > 0$, while $(d/dx)x = 1$. To find the graph of the inverse function f^{-1}, we then just reflect the graph of f across the line $y = x$.

A function and its inverse are not only related with respect to their graphs, but also with respect to their derivatives, as we now show.

Theorem 2.21 (Inverse Differentiation). *Let f be a function of a real variable with inverse $g = f^{-1}$. If x is in the domain of g, and if $f'(g(x)) \neq 0$, then $g'(x)$ exists, and*

$$g'(x) = \frac{1}{f'(g(x))}.$$

Remark 2.22. The proof of this identity follows from the Chain Rule. If we additionally assume that g is differentiable at x, then the Chain rule applied to $f(g(x)) = x$ says that

$$f'(g(x)) \cdot g'(x) = \frac{d}{dx} f(g(x)) = \frac{d}{dx} x = 1.$$

That is, $g'(x) = 1/(f'(g(x)))$.

Remark 2.23. This Theorem is another tool that will allow us to understand the inverse of the exponential function. In particular, it will allow us to compute the derivative of the inverse of the exponential function.

Example 2.24. Let's consider again the function $f(x) = x^3$ on $(-\infty, \infty)$, where $f^{-1}(x) = x^{1/3}$. Then $f'(x) = 3x^2$, so

$$(f^{-1})'(x) = [f'(f^{-1}(x))]^{-1} = [3(f^{-1}(x))^2]^{-1} = x^{-2/3}/3.$$

However, we already "knew" that $(f^{-1})'(x) = (1/3)x^{-2/3}$, by our usual differentiation rules.

Remark 2.25. You should not confuse the notation f^{-1}, which denotes the inverse function of f, with a number to the -1 power. That is, it is typically true that $f^{-1}(x) \neq (f(x))^{-1}$.

2.3. **Logarithms.** As discussed in Remark 2.18, if $b > 0$ with $b \neq 1$, then the function $f(x) = b^x$ is invertible. We write $f^{-1}(x) = \log_b(x)$. By the definition of invertibility, we have

Proposition 2.26.

$$b^{\log_b x} = x, \qquad \log_b b^x = x.$$

Remark 2.27. When $b = e$, we write $\log_b(x) = \ln(x)$, and we call $\ln(x)$ the **natural logarithm**.

Example 2.28. Since $2^3 = 8$, we have $\log_2(8) = \log_2(2^3) = 3$, and $2^{\log_2 8} = 8$.
Since $3^{-2} = 1/9$, we have $\log_3(1/9) = \log_3(3^{-2}) = -2$, and $3^{\log_3(1/9)} = 1/9$.

Let $b > 0$ with $b \neq 1$. Let $f(x) = b^x$. If $b > 1$, then

$$\lim_{x\to\infty} b^x = \infty, \qquad \lim_{x\to-\infty} b^x = \lim_{x\to\infty} \frac{1}{b^x} = 0.$$

So, f has domain $(-\infty, \infty)$ and range $(0, \infty)$. So, if $b > 1$, we can see from the graph of $\log_b$ that

$$\lim_{x\to\infty} \log_b(x) = \infty, \qquad \lim_{x\to 0^+} \log_b(x) = -\infty.$$

If $b < 1$, then

$$\lim_{x\to\infty} b^x = 0, \qquad \lim_{x\to-\infty} b^x = \lim_{x\to\infty} \frac{1}{b^x} = \infty.$$

So, f has domain $(-\infty, \infty)$ and range $(0, \infty)$. And, if $b < 1$, we can see from the graph of $\log_b$ that

$$\lim_{x\to\infty} \log_b(x) = -\infty, \qquad \lim_{x\to 0^+} \log_b(x) = +\infty.$$

Remark 2.29. The logarithm of a negative number is undefined. To see why, note that if we could write $x = \log_3(-2)$, then by exponentiation both sides, we would have $3^x = -2$. But this equation has no real solution.

Here is a summary of properties of logarithm functions. These properties typically follow by the corresponding properties of the exponential functions

Proposition 2.30 (Properties of Logarithm Functions). *Let $b > 0$, with $b \neq 1$, let x, y be positive real numbers, let n be a positive integer, and let $f(x) = \log_b(x)$.*

- *The domain of $f(x) = \log_b(x)$ is the set of all positive numbers, and the range of f is the set of all real numbers*
- *If $b > 1$, then $f(x) = \log_b(x)$ is increasing. If $0 < b < 1$, then f is decreasing.*
- $\log_b(1) = 0$.
- $\log_b(xy) = \log_b(x) + \log_b(y)$
- $\log_b(x/y) = \log_b(x) - \log_b(y)$.
- $\log_b(1/x) = -\log_b(x)$.
- $\log_b(x^n) = n \log_b x$.

Moreover, if $a > 0$ with $a \neq 1$, then the logarithm functions are proportional in the following sense

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}, \qquad \log_b(x) = \frac{\ln x}{\ln b}$$

To see why some of these properties are true, recall that $b^0 = 1$. So, taking the log of both sides, $\log_b(1) = 0$, proving the third property. To prove the fourth property, note that

$$b^{\log_b(xy)} = xy = (b^{\log_b(x)})(b^{\log_b(y)}) = b^{\log_b(x)+\log_b(y)}.$$

So, taking $\log_b$ of both sides, we get $\log_b(xy) = \log_b(x) + \log_b(y)$. The other properties follow in a similar way.

To see the final property, note that

$$e^{\ln(b)\log_b(x)} = (e^{\ln(b)})^{\log_b(x)} = b^{\log_b(x)} = x.$$

So, taking ln of both sides shows that $\ln(b)\log_b(x) = \ln(x)$, proving the final property. This property then implies that

$$\log_b(x) = \frac{\ln x}{\ln b} = \frac{(\ln x)/(\ln a)}{(\ln b)/(\ln a)} = \frac{\log_a(x)}{\log_a(b)}.$$

Example 2.31.

$$\log_6(9) + \log_6(4) = \log_6(9 \cdot 4) = \log_6(36) = \log_6(6^2) = 2.$$

Using the properties of the logarithm, we can now finally differentiate exponential functions and logarithms.

Theorem 2.32 (Derivative of Exponential). *Let $b > 0$. Then*

$$\frac{d}{dx} b^x = (\ln b) b^x.$$

This formula follows readily from the properties of the logarithm, as follows.

$$\frac{d}{dx} b^x = \frac{d}{dx}(e^{\ln b})^x = \frac{d}{dx} e^{x \ln b} = (\ln b) e^{x \ln b} = (\ln b)(e^{\ln b})^x = (\ln b) b^x.$$

We can then use the inverse differentiation formula to differentiate the logarithm as follows.

Theorem 2.33 (**Derivative of Logarithm**). *Let $x > 0$. Then*

$$\frac{d}{dx} \ln x = \frac{1}{x}.$$

Consequently, for any $b > 0$ with $b \neq 1$, since $\log_b(x) = \frac{\ln x}{\ln b}$, we have

$$\frac{d}{dx} \log_b(x) = \frac{1}{x \ln b}.$$

To justify this differentiation, we apply the inverse differentiation formula (Theorem 2.21) where $f(x) = e^x$ and $f^{-1}(x) = \ln x$ to get

$$\frac{d}{dx} \ln x = \frac{1}{f'(f^{-1}(x))} = \frac{1}{e^{f^{-1}(x)}} = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

Remark 2.34. Let f be a positive function. From the chain rule, we have

$$\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}.$$

This formula can be useful for calculating derivatives.

Example 2.35. Let j be a positive number. Let's calculate the derivative of

$$f(x) = x^j.$$

Strictly speaking, you never learned why the formula $f'(x) = jx^{j-1}$ holds. Let's see why it holds now. Let $x > 0$. Then $\ln f(x) = j \ln x$. So, differentiating both sides and using the above remark,

$$\frac{f'(x)}{f(x)} = \frac{j}{x}.$$

That is, $f'(x) = \frac{f(x)j}{x} = jx^{j-1}$.

We can carry this method further to more exotic functions that are not covered by any rules we have learned before. Consider

$$f(x) = x^x.$$

You might think the Chain Rule applies, but I do not see an obvious method for applying the chain rule. You can use some properties of the logarithm function to write

$$\frac{d}{dx} x^x = \frac{d}{dx} e^{x \ln x} = e^{x \ln x} \frac{d}{dx}(x \ln x) = e^{x \ln x}(1 + \ln x) = x^x(1 + \ln x).$$

Alternately, we can write

$$\frac{f'(x)}{f(x)} = \frac{d}{dx} \ln(x^x) = \frac{d}{dx}(x \ln x) = 1 + \ln x.$$

So that, once again, $f'(x) = f(x)(1 + \ln x) = x^x(1 + \ln x)$.

Remark 2.36. Since $(d/dt) \ln t = 1/t$ for $t > 0$ and $\ln(1) = 0$, we have the following formula for the natural logarithm. Let $x > 0$. Then from the fundamental theorem of calculus,

$$\ln(x) = \int_1^x \frac{1}{t} dt.$$

Remark 2.37. Note that for $t < 0$, $(d/dt)\ln(-t) = 1/t$, so for any $t \neq 0$, we have

$$\frac{d}{dt}\ln|t| = \frac{1}{t}$$

So, from the chain rule, if f is a nonzero function, then

$$\frac{d}{dt}\ln|f(t)| = \frac{f'(t)}{f(t)}.$$

Example 2.38.

$$\int_3^9 \frac{dt}{t} = \ln(9) - \ln(3) = \ln(9/3) = \ln 3.$$

Exercise 2.39. Evaluate

$$\int_{-9}^{-3} \frac{1}{t}dt.$$

Example 2.40. Now we can compute some antiderivatives that we could not compute before. Consider the function $f(x) = \frac{x}{x^2+1}$. This function is of the form $(1/2)g'(x)/g(x)$ where $g(x) = x^2 + 1$. So, an antiderivative for f is $(1/2)\ln|x^2 + 1|$.

Consider also the function $f(x) = \tan(x) = \sin(x)/\cos(x)$. This function is of the form $-g'(x)/g(x)$ where $g(x) = \cos(x)$. So, an antiderivative for f is $-\ln|\cos(x)|$.

2.4. **Applications: exponential growth, compound interest.** One of the many reasons to care about the exponential function is that it appears in so many applications. For example, bacteria grow at an exponential rate (if they have unlimited food and space). Money in a bank account grows at an exponential rate. But also, money devalues at an exponential rate (roughly three percent per year in the US.) This last observation (known as inflation) explains why most common material possessions "used to cost a nickel and a dime." Specifically, if money loses roughly three percent of its value per year, then the current value of one dollar will be equal to the value of two dollars in roughly twenty years. Put another way, the amount of goods you can buy with a fixed amount of money gets cut in half, roughly every twenty years. So, one hundred years ago, money was roughly $2^5 = 32$ times more valuable. A dime roughly a hundred years ago could buy the same amount as around three dollars now. A $20,000$ dollar car now would cost roughly 600 dollars one hundred years ago. And indeed, the latter number is comparable to the original price of a Ford Model T. To compare money at different historical periods, we therefore hear monetary amounts that have been inflation adjusted, which means that they take into account this change in the value of money over time.

Let's be more precise about exponential growth. Let t, k and P_0 be real numbers, and define

$$P(t) = P_0 e^{kt}.$$

If $k > 0$, we say that P **grows exponentially**. If $k < 0$, we say that P **decays exponentially**.

For example, if $P(t)$ denotes a population of bacteria at time t, then $P(0) = P_0$ is the population of the bacteria at time zero, and $k > 0$ represents the rate of growth of the bacteria.

Note that $P(t)$ satisfies the differential equation

$$P'(t) = kP(t), \qquad P(0) = P_0.$$

It turns out that this differential equation uniquely characterizes the exponential function.

Theorem 2.41 (Differential Equation Characterization of Exponential). *Let $y(t)$ be a continuously differentiable function of t such that*

$$y'(t) = ky(t), \qquad y(0) = P_0.$$

Then we must have $y(t) = P_0 e^{kt}$.

To justify this statement, we differentiate $y(t)e^{-kt}$ to get

$$\frac{d}{dt}(y(t)e^{-kt}) = -ky(t)e^{-kt} + y'(t)e^{-kt} = e^{-kt}(-ky(t) + ky(t)) = 0.$$

Therefore, there exists a constant C such that $y(t)e^{-kt} = C$, so that $y(t) = Ce^{kt}$. Since $P_0 = y(0) = Ce^0 = C$, we conclude that $C = P_0$, so that $y(t) = P_0 e^{kt}$, as desired.

Remark 2.42. We interpret this characterization of the exponential function as follows. Suppose we have a quantity $y(t)$ whose growth is proportional to $y(t)$. Then $y(t)$ must be an exponential function. In this way, the exponential function arises naturally when discussing the growth of bacteria, interest rates, radioactive decay, chemical kinetics, and so on. However, more complicated differential equations have more exotic solutions.

Example 2.43. Certain therapeutic drugs in the body are filtered by the kidneys at a rate which is proportional to their concentration in the blood stream. That is, if $y(t)$ is the concentration of the drug at time t, then there exists a constant $k < 0$ such that $dy/dt = ky$. Therefore, $y(t) = y_0 e^{kt}$.

Example 2.44. In chemistry class, we learned about "first order" reactions. In these reactions, the concentration $y(t)$ of a certain chemical is proportional to its rate of change. So, as before, $dy/dt = ky(t)$ and then $y(t) = y_0 e^{kt}$.

Exercise 2.45. In chemistry, a "second order" reaction satisfies $dy/dt = k(y(t))^2$. If $y_0 = y(0) \neq 0$, verify that we must have

$$y(t) = \frac{1}{y_0^{-1} - kt}.$$

Example 2.46. Radioactive isotopes decay at a rate that is proportional to their concentration.

As we discussed in the introduction, it is helpful to think about the doubling time (if $k > 0$) or half-life (if $k < 0$) of an exponential function $P(t) = P_0 e^{kt}$. Given any real t, the **doubling time** $T > 0$ of $P(t) = P_0 e^{kt}$ is a time such that

$$\begin{cases} P(t+T) = 2P(t) & \text{, if } k > 0 \\ P(t+T) = (1/2)P(t) & \text{, if } k < 0 \end{cases}.$$

It is evident that we have

$$T = \frac{\ln 2}{|k|}.$$

To see this, note that

$$P\left(t + \frac{\ln 2}{|k|}\right) = P(t)e^{\frac{k \ln 2}{|k|}} = P(t)2^{k/|k|} = \begin{cases} 2P(t) & \text{, if } k > 0 \\ (1/2)P(t) & \text{, if } k < 0 \end{cases}.$$

Example 2.47. A radioactive isotope decays exponentially with a half-life of 4 days. Suppose we want to know how long it takes for 80% of the isotope to decay. In this case, we have $P(t) = P_0 e^{-kt}$ where $k > 0$, and the half life satisfies $4 = (\ln 2)/k$, so that $k = (\ln 2)/4$. That is,

$$P(t) = P_0 e^{-t(\ln 2)/4} = \frac{P_0}{2^{t/4}}.$$

We want to find t such that $P(t)/P_0 = .2$. That is, we need to solve $2^{-t/4} = .2$, i.e. $(-t/4)\ln 2 = \ln(1/5)$, so $t = \frac{4\ln 5}{\ln 2} \approx 9.2877$. Note that this answer is reasonable since the doubling time is 4 days, so it takes 8 days to get 75% decay, so 80% decay should take a bit longer than 8 days (but not more than 12, which would have 87.5% decay).

2.5. **Compound Interest.** As we discussed above, if I have an account that starts with P_0 dollars at time $t = 0$ which earns an annual interest rate r which is compounded M times during the year, then the amount of money in the account after t years is

$$P(t) = P_0 \left(1 + \frac{r}{M}\right)^{Mt}.$$

Also, using the limit formula for e

$$e^r = \lim_{M\to\infty} \left(1 + \frac{r}{M}\right)^{M},$$

we see that as the compounding time per year M goes to infinity, we actually reach some finite limiting quantity. That is,

$$\lim_{M\to\infty} P(t) = P_0 \left[\lim_{M\to\infty} \left(1 + \frac{r}{M}\right)^{M}\right]^t = P_0(e^r)^t = P_0 e^{rt}.$$

We therefore say that, if an interest rate is **compounded continuously** at the rate r, then

$$P(t) = P_0 e^{rt}.$$

In fact, such a consideration led to the very discovery of the constant e.

We will now return to some of our economic discussions related to interest rates and deflation.

Definition 2.48. Suppose there is a (continuously compounded) interest rate $r > 0$ at which I can lend or borrow money. The **present value** of an amount P of money that is received at time t is defined as Pe^{-rt}.

Example 2.49. If $r = .1$, and if you agree to give me $1000 twenty years from now, then this would be equivalent to giving me $1000e^{-.1(20)} = 1000e^{-2} \approx \$135$ right now (since I could put this money into an account that earns 10% interest for twenty years.)

Example 2.50. Returning to our discussion of inflation, if $r = .03$ is the rate of inflation of money, then money becomes half as valuable roughly every $(\ln 2)/.03 \approx 23$ years. So, if you agree to buy me a $20000 car in 115 years, this would be equivalent to buying a car worth $20000e^{-.03(115)} \approx 635$ dollars right now.

To some extent, we can think of an income as a continuous function $R(t)$, in which case we can compute the present value of that income as follows.

Definition 2.51. Let $r > 0$ be an interest rate, let $R(t)$ be an income stream that pays $R(t)$ per year continuously for T years. Then the present value of this income stream is

$$\int_0^T R(t)e^{-rt}dt.$$

Example 2.52. Suppose I am earning \$10000 per year continuously for 5 years, and once I get the money, I put it into an interest bearing account with interest rate $r = .03$. We therefore have $R(t) = 10000$, and the present value of this income is

$$\int_0^5 10000e^{-.03t}dt = 10000\frac{1}{-.03}[e^{-.03t}]_{t=0}^{t=5} = 10000\frac{1}{-.03}[e^{-.15} - 1] \approx 46431.$$

That is, it would be equivalent for me to get \$46431 right now and put it into the interest bearing account, rather than continuously putting the money into the account.

2.6. **Terminal Velocity.** Suppose an object of mass m is falling due to the force of gravity $-mg$ in the negative y-direction. The force due to air friction is then $-kv$ where v is the velocity of the object and $k > 0$. That is, the air friction opposes the velocity of the object. The total force on the object is then $-mg - kv$. If a denotes the acceleration of the object, then $a = v'$ and $F = ma$, so $ma = mv' = -mg - kv$. That is, $v'(t) = -g - kv(t)/m$. That is,

$$v'(t) = -g - kv(t)/m = (-k/m)(v(t) + gm/k).$$

If $f(t) = v(t) + gm/k$, then $f'(t) = v'(t) = (-k/m)f(t)$, so $f(t) = Ce^{-kt/m}$. That is,

$$v(t) = Ce^{-kt/m} - gm/k.$$

As $t \to \infty$, we see that

$$\lim_{t\to\infty} v(t) = -gm/k.$$

We therefore refer to $-gm/k$ as the **terminal velocity** of the body. That is, after falling for a while, the forces of gravity and air friction cancel each other out, and the body falls at a constant velocity (the terminal velocity).

A similar differential equation gives Newton's Law of Cooling.

Exercise 2.53 (Newton's Law of Cooling). Suppose $y(t)$ is the temperature of an object at time t. If an object is of a different temperature than its surroundings, then the rate of change of the object's temperature is proportional to the difference of the temperature of the object and the temperature of the surroundings. That is, if Y denotes the temperature of the surroundings, and if $y(0) = y_0 \neq Y$, then there exists a constant $k > 0$ such that

$$y'(t) = -k(y(t) - Y).$$

Note that if $y(0) < Y$, then $y'(0) > 0$, so that the temperature of y is increasing to the environment's temperature. And if $y(0) > Y$, then $y'(0) < 0$, so that y is decreasing to the environment's temperature.

Let $f(t) = y(t) - Y$. Verify that $f'(t) = -kf(t)$. Conclude that $f(t) = y(t) - Y = (y_0 - Y)e^{-kt}$. That is, we have Newton's Law of cooling:

$$y(t) = Y + (y_0 - Y)e^{-kt}.$$

Exercise 2.54. The exponential growth model for bacteria is a bit unrealistic, since after a while, the bacteria are limited by their environment and food supply. We therefore consider the **logistic growth** model. Suppose $y(t)$ is the amount of bacteria in a petri dish at time t and $k > 0$ is a constant. Let C be the maximum possible population of the bacteria. We model the growth of the bacteria by the formula

$$y'(t) = ky(t)(C - y(t)), \qquad y(0) = y_0$$

So, when y is small, $y'(t)$ is proportional to y. However, when y becomes close to C, y' becomes very small. That is, the rate of growth of bacteria is constrained by the environment.

- Verify that the following function satisfies the above differential equation.

$$y(t) = \frac{C}{1 + (C^{-1}y_0^{-1} - 1)e^{-kt}}.$$

- Plot the function $y(t)$. (What are the limits of y as t goes to $+\infty$ and $-\infty$?)
- Find out where $y'(t)$ is the largest. (Hint: find the maximum of the function of y: $ky(C - y)$.)

The latter observation explains the "J-curve" scare for human population growth in the 1980s. At this point in time, many people were afraid that the human population would grow too large for the earth to support us. However, it seems that we were simply observing the maximum possible growth rate of the human population at this time (if we believe that logistic growth models the human population reasonably well).

3. L'Hopital's Rule

Having satisfied our thirst for applications, we now veer back towards some more theoretical results that find many uses. The first such result is L'Hopital's rule. This rule allows the computation of many limits of functions that we were unable to handle before.

We will first state this rule, and then give several examples of its application.

Theorem 3.1 (L'Hôpital's Rule). *Let f, g be differentiable functions on an open interval containing a. (We allow f, g to not be differentiable at a.) Assume that $g'(x) \neq 0$ except possibly at a. Assume that*

$$f(a) = g(a) = 0.$$

If $\lim_{x\to a} f'(x)/g'(x)$ exists or is $+\infty$ or is $-\infty$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

This conclusion also holds if, instead of assuming $f(a) = g(a) = 0$, we assume

$$\lim_{x\to a} f(x) = \pm\infty, \quad \textit{and} \quad \lim_{x\to a} g(x) = \pm\infty.$$

Remark 3.2. L'Hôpital's Rule remains true if we replace all limits above with right-sided limits $\lim_{x\to a^+}$. Similarly, this rule remains true if we replace all limits above with left-sided limits $\lim_{x\to a^-}$.

Example 3.3.

$$\lim_{x\to 0} \frac{\sin x}{x} \overset{L'H}{=} \lim_{x\to 0} \frac{\cos x}{1} = 1.$$

So, $\sin(x) \approx x$ when x is near zero.

Example 3.4.

$$\lim_{x\to 0}\frac{(\cos x)-1}{x^2}\overset{L'H}{=}\lim_{x\to 0}\frac{-\sin x}{2x}\overset{L'H}{=}\lim_{x\to 0}\frac{-\cos x}{2}=-1/2.$$

So, $\cos(x)\approx 1-x^2/2$ when x is near zero.

Example 3.5. Sometimes we have to change our function around a bit before we can apply L'Hôpital's rule. For example, as $x\to 0^+$, the function $x\ln x$ looks like 0 multiplied by $-\infty$, so L'Hôpital's rule does not directly apply. We therefore write

$$x\ln x=\frac{\ln x}{1/x}.$$

This way, both the top and bottom go to $\pm\infty$ as $x\to 0^+$, so we can apply L'Hôpital's rule.

$$\lim_{x\to 0^+}x\ln x=\lim_{x\to 0^+}\frac{\ln x}{x^{-1}}\overset{L'H}{=}\lim_{x\to 0^+}\frac{x^{-1}}{-x^{-2}}=\lim_{x\to 0^+}(-x)=0.$$

Here are some other examples where we need to change our function around a bit before applying L'Hôpital's rule.

Example 3.6.

$$\lim_{x\to 0^+}x^x=\lim_{x\to 0^+}e^{x\ln x}=e^{\lim_{x\to 0^+}x\ln x}=e^0=1.$$

Example 3.7.

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\lim_{x\to 0}\frac{x-\sin x}{x\sin x}\overset{L'H}{=}\lim_{x\to 0}\frac{1-\cos x}{x\cos x+\sin x}\overset{L'H}{=}\lim_{x\to 0}\frac{\sin x}{-x\sin x+2\cos x}=\frac{0}{2}=0.$$

Example 3.8. We need to know that the assumptions of L'Hôpital's rule apply before we use this rule. For example,

$$\lim_{x\to 0}\frac{x+2}{x+1}=2.$$

However, $\lim_{x\to 0}\frac{(d/dx)(x+2)}{(d/dx)(x+1)}=1\neq 2$.

3.1. **Growth of Functions.** L'Hôpital's rule also allows us to understand the growth of functions at infinity. For example, in this section, we will explain the heuristic that exponential growth is "very fast" and logarithmic growth is "very slow."

Definition 3.9. Let f,g be positive functions. We say that f **grows faster than** g at ∞ if

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\infty$$

Or equivalently,

$$\lim_{x\to\infty}\frac{g(x)}{f(x)}=0.$$

Example 3.10. x^2 grows faster than x as $x\to\infty$, since $\lim_{x\to\infty}\frac{x^2}{x}=\lim_{x\to\infty}x=\infty$.

Remark 3.11. We can apply L'Hôpital's rule for limits involving infinity. Also, we can apply L'Hôpital's rule if both functions go to $\pm\infty$ as $x\to\infty$.

Example 3.12. We show that e^x grows faster than x^3 at ∞.

$$\lim_{x\to\infty} \frac{e^x}{x^3} \overset{L'H}{=} \lim_{x\to\infty} \frac{e^x}{3x^2} \overset{L'H}{=} \lim_{x\to\infty} \frac{e^x}{6x} \overset{L'H}{=} \lim_{x\to\infty} \frac{e^x}{6} = \infty.$$

More generally, e^x grows faster than any polynomial at ∞.

Example 3.13. Note also that e^{-x} decays faster than any function x^{-j} with $j > 0$ at ∞.

Example 3.14. The logarithm grows slower than x^j for any $j > 0$ at ∞.

$$\lim_{x\to\infty} \frac{\ln x}{x^j} \overset{L'H}{=} \lim_{x\to\infty} \frac{x^{-1}}{jx^{j-1}} = \lim_{x\to\infty} j^{-1}x^{-j} = 0.$$

4. Inverse Trigonometric Functions

In order to broaden our vocabulary of functions, we will now discuss some of the inverses of trigonometric functions. The functions sin, cos and tan all fail the horizontal line test, so we need to restrict their domains to define their inverses. Here are the standard ways of doing so

Definition 4.1. We restrict $\sin\theta$ to the domain $[-\pi/2, \pi/2]$. Then sin is invertible with range $[-1, 1]$, so we define $\sin^{-1}$ to be this inverse function. Note that $\sin^{-1}$ has domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$.

Example 4.2. Since $\sin(\pi/2) = 1$, $\sin(0) = 0$, $\sin(\pi/4) = \sqrt{2}/2$, we have

$$\sin^{-1}(1) = \pi/2, \qquad \sin^{-1}(0) = 0, \qquad \sin^{-1}(\sqrt{2}/2) = \pi/4.$$

Definition 4.3. We restrict $\cos\theta$ to the domain $[0, \pi]$. Then cos is invertible with range $[-1, 1]$, so we define $\cos^{-1}$ to be this inverse function. Note that $\cos^{-1}$ has domain $[-1, 1]$ and range $[0, \pi]$.

Example 4.4. Since $\cos(0) = 1$, $\cos(\pi/2) = 0$, $\cos(\pi) = -1$, we have

$$\cos^{-1}(1) = 0, \qquad \cos^{-1}(0) = \pi/2, \qquad \cos^{-1}(-1) = \pi.$$

Definition 4.5. We restrict $\tan\theta$ to the domain $(-\pi/2, \pi/2)$. Then tan is invertible with range $(-\infty, \infty)$, so we define $\tan^{-1}$ to be this inverse function. Note that $\tan^{-1}$ has domain $(-\infty, \infty)$ and range $(-\pi/2, \pi/2)$.

Example 4.6. Since $\tan(0) = 0$, $\tan(\pi/4) = 1$, $\lim_{x\to\pi/2^-} \tan(x) = \infty$, we have

$$\tan^{-1}(0) = 0, \qquad \tan^{-1}(1) = \pi/4, \qquad \lim_{x\to\infty} \tan^{-1}(x) = \pi/2.$$

Example 4.7. The inverse derivative formula also allows us to compute the derivatives of inverse trigonometric functions. Consider $f(x) = \tan(x) = \sin(x)/\cos(x)$. Recall that $f'(x) = 1/\cos^2(x)$. Let x be a real number. If $x = f(y) = \tan(y)$, then $f^{-1}(x) = y$. Recall that the tangent function has input y which is an angle of a right triangle, and it has output the ratio of edges of a right triangle. So, the inverse tangent has input the ratio of edges of a right triangle and output the angle of the triangle. So, if a right triangle has height x and width 1, then $\tan^{-1}(x)$ is the angle of the triangle (which has hypotenuse $\sqrt{1+x^2}$, by the Pythagorean Theorem.) Then, the cosine of the angle $\tan^{-1}(x)$ is $1/\sqrt{1+x^2}$. That is, $\cos(\tan^{-1}(x)) = 1/\sqrt{1+x^2}$. Finally, applying our differentiation formula,

$$(\tan^{-1}(x))' = [f'(f^{-1}(x))]^{-1} = \cos^2(f^{-1}(x)) = \cos^2(\tan^{-1}(x)) = \frac{1}{|1+x^2|} = \frac{1}{1+x^2}.$$

Similarly, we have

$$\cos(\sin^{-1}(x)) = \sqrt{1-x^2}.$$

$$\sin(\cos^{-1}(x)) = \sqrt{1-x^2}.$$

So, using the inverse derivative formula, we get

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\cos(\sin^{-1}(x))} = \frac{1}{\sqrt{1-x^2}}.$$

$$\frac{d}{dx}\cos^{-1}(x) = \frac{1}{-\sin(\cos^{-1}(x))} = -\frac{1}{\sqrt{1-x^2}}.$$

4.1. **Hyperbolic Functions.** The hyperbolic functions are closely related to their trigonometric counterparts, though we cannot explain exactly why in this course. Let's define hyperbolic sine and hyperbolic cosine as follows.

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}.$$

The other hyperbolic functions are then defined in a similar way to trigonometric functions.

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}, \qquad \mathrm{sech}(x) = \frac{1}{\cosh(x)}, \qquad \coth(x) = \frac{1}{\tanh(x)}, \qquad \mathrm{csch}(x) = \frac{1}{\sinh(x)}.$$

We can verify directly that

$$\sinh'(x) = \cosh(x), \qquad \cosh'(x) = \sinh(x).$$

$$\sinh^2(x) = \cosh^2(x) - 1.$$

The hyperbolic functions are invertible on the domain $(-\infty, \infty)$, except for cosh and sech. We restrict the domains of cosh and sech to $x \geq 0$ to define their inverses. I would only like to derive one such inverse derivative; that of sech^{-1}. Since sech has domain $[0, \infty)$ and range $(0, 1]$, we see that sech^{-1} has domain $(0, 1]$ and range $[0, \infty)$. So, let $0 < x \leq 1$. Note that

$$\frac{d}{dx}\mathrm{sech}(x) = -\frac{\cosh'(x)}{\cosh(x)^2} = -\frac{\sinh(x)}{\cosh(x)^2}.$$

$$\mathrm{sech}^{-1}(x) = \cosh^{-1}(1/x), \quad \text{since} \quad \mathrm{sech}(\cosh^{-1}(1/x)) = \frac{1}{\cosh(\cosh^{-1}(1/x))} = x.$$

$$\cosh(\mathrm{sech}^{-1}(x)) = \cosh(\cosh^{-1}(1/x)) = 1/x.$$

$$\sinh(\mathrm{sech}^{-1}(x)) = \sinh(\cosh^{-1}(1/x)) = \sqrt{\cosh^2(\cosh^{-1}(1/x)) - 1} = \sqrt{x^{-2} - 1}.$$

Therefore, from the inverse derivative formula, if $0 < x \leq 1$, we have

$$\frac{d}{dx}\mathrm{sech}^{-1}(x) = -\frac{\cosh(\mathrm{sech}^{-1}(x))^2}{\sinh(\mathrm{sech}^{-1}(x))} = \frac{-x^{-2}}{\sqrt{x^{-2}-1}} = -\frac{1}{x\sqrt{x^2(x^{-2}-1)}} = -\frac{1}{x\sqrt{1-x^2}}.$$

5. Methods of Integration

We now turn our attention to various methods that help to compute integrals. By the end of the chapter, you should feel like you can integrate almost anything.

5.1. Integration by Parts.

This first method of integration allows us to essentially move a derivative from one term to another while we are inside an integral.

Theorem 5.1 (Integration by Parts). *Let u, v be continuously differentiable functions on the real line. Let $a < b$. Then*

$$\int_a^b u(x)v'(x)dx = u(b)v(b) - u(a)v(a) - \int_a^b v(x)u'(x)dx.$$

This rule can be memorized by the mnemonic "integral of udv equals uv minus integral of vdu."

The integration by parts formula follows almost immediately from the product rule.

$$(uv)' = u'v + v'u.$$

Integrating both sides on $[a, b]$ and applying the Fundamental Theorem of Calculus to the left side,

$$u(b)v(b) - u(a)v(a) = \int_a^b u'(x)v(x)dx + \int_a^b u(x)v'(x)dx.$$

Example 5.2.

$$\int_a^b x\cos x dx = \int_a^b x(d/dx)\sin x dx = b\sin b - a\sin a - \int_a^b \sin(x)dx$$
$$= b\sin b - a\sin a + \cos b - \cos a.$$

Written another way,

$$\int x\cos x dx = x\sin x + \cos x + C$$

Remark 5.3. If we instead wrote $\int_a^b x\cos x dx = \int_a^b \cos x(d/dx)(x^2/2)dx$, then things would have just become more complicated. So, we have to choose carefully how to apply integration by parts.

Example 5.4.

$$\int_1^4 \ln x\, dx = \int_1^4 \ln x(d/dx)x dx = 4\cdot\ln 4 - 1\cdot\ln 1 - \int_1^4 x(1/x)dx = 4\ln 4 - 1\ln 1 - 4.$$

Written another way,

$$\int \ln x\, dx = x\ln x - x + C$$

Example 5.5.

$$\int_a^b \cos^3(x)dx = \int_a^b \cos^2(x)(d/dx)(\sin x)dx = [\cos^2(x)\sin(x)]_{x=a}^{x=b} + \int_a^b 2\sin^2(x)\cos(x)dx$$

$$= [\cos^2(x)\sin(x)]_{x=a}^{x=b} + \int_a^b 2(1-\cos^2(x))\cos(x)dx$$

$$= [\cos^2(x)\sin(x)]_{x=a}^{x=b} + 2\int_a^b \cos(x)dx - 2\int_a^b \cos^3(x)dx.$$

So, moving the last term to the left side,

$$3\int_a^b \cos^3(x)dx = [\cos^2(x)\sin(x)]_{x=a}^{x=b} + 2[\sin(x)]_{x=a}^{x=b}.$$

That is,

$$\int \cos^3(x)dx = (1/3)\cos^2(x)\sin(x) + (2/3)\sin(x) + C.$$

5.2. **Trigonometric Integrals.** In this section, we will see how to use integration by parts and substitution to compute integrals of the form

$$\int \sin^m(x)\cos^n(x)dx, \qquad n, m \text{ nonnegative integers.}$$

Example 5.6 (m **or** n **is odd**). Suppose for example that m is odd. We demonstrate the case $m = 5$, $n = 4$. The idea is that we split off one factor of sin, and then write everything else in terms of cosines.

$$\int \sin^5(x)\cos^4(x)dx = \int \sin^4(x)\cos^4(x)\sin(x)dx = \int (1-\cos^2(x))^2\cos^4(x)\sin(x)dx$$

$$= -\int (1-u^2)^2 u^4 du = (1/5)u^5 - (2/7)u^7 + (1/9)u^9$$

$$= (1/5)\cos^5(x) - (2/7)\cos^7(x) + (1/9)\cos^9(x).$$

The same procedure works when n is odd. In this case, we split off one factor of cos, and then we write everything else in terms of sines, and so on.

The only remaining case occurs when both n and m are even. In this case, using $\sin^2(x) + \cos^2(x) = 1$, it suffices to consider the case that $m = 0$ and n is even (though the case $n = 0$ and m even is treated similarly).

Example 5.7 (m **is zero and** n **is even**). We demonstrate this case by taking $n = 6$, and showing that this reduces to the case $n = 4$. In general, given any even n, we can reduce to the case $n - 2$.

$$\int \cos^6(x)dx = \int \cos^5(x)(d/dx)\sin(x)dx = \cos^5(x)\sin(x) + 5\int \cos^4(x)\sin^2(x)dx$$

$$= \cos^5(x)\sin(x) + 5\int \cos^4(x)(1-\cos^2(x))dx$$

$$= \cos^5(x)\sin(x) - 5\int \cos^6(x)dx + 5\int \cos^4(x)dx$$

So, adding the middle term from both sides,

$$6\int \cos^6(x)dx = -\cos^5(x)\sin(x) + 5\int \cos^4(x)dx.$$

Finally, in the case $n = 2$, we simply use the identity

$$\cos^2(x) = (1/2)(1+\cos(2x)), \qquad \sin^2(x) = (1/2)(1-\cos(2x)).$$

Example 5.8 ($n = 2$, $m = 0$)**.**

$$\int \cos^2(x) = (1/2)\int (1+\cos(2x))dx = (1/2)(x + (1/2)\sin(2x)) + C.$$

5.3. **Trigonometric Substitution.** We will now show how to use substitution of trigonometric expressions to evaluate some integrals involving the square root function:

$$\sqrt{a^2 - x^2}, \qquad \sqrt{x^2 + a^2}, \qquad \sqrt{x^2 - a^2}.$$

Example 5.9. Let $-1 \leq x \leq 1$. In this example, we use the substitution $x = \sin\theta$, where $-\pi/2 \leq \theta \leq \pi/2$. In this region, $\cos\theta \geq 0$, so we make use of the fact that $\sqrt{\cos^2\theta} = \cos\theta$ for such θ.

$$\begin{aligned}
\int \sqrt{1-x^2}\,dx &= \int \sqrt{1-\sin^2\theta}\cos\theta d\theta = \int \sqrt{\cos^2\theta}\cos\theta d\theta \\
&= \int \cos^2(\theta)d\theta = \int (1/2)(1+\cos(2\theta))d\theta \\
&= (1/2)[\theta + (1/2)\sin(2\theta)] + C = (1/2)[\sin^{-1}(x) + (1/2)\sin(2\sin^{-1}(x))] + C
\end{aligned}$$

We can simplify this expression by noting that $\sin(2y) = 2\sin(y)\cos(y)$, so

$$\sin(2\sin^{-1}(x)) = 2\sin(\sin^{-1}(x))\cos(\sin^{-1}(x)).$$

Since $-1 \leq x \leq 1$, $\sin(\sin^{-1}(x)) = x$. We now compute $\cos(\sin^{-1}(x))$. If $y = \sin^{-1}(x)$, then y is the angle of a right triangle with height x and hypotenuse 1. So, this right triangle has base $\sqrt{1-x^2}$, by the Pythagorean Theorem. Therefore, $\cos(y) = \sqrt{1-x^2}$. That is,

$$\cos(\sin^{-1}(x)) = \sqrt{1-x^2}.$$

In conclusion,

$$\int \sqrt{1-x^2}\,dx = (1/2)\sin^{-1}(x) + (1/2)x\sqrt{1-x^2} + C.$$

Remark 5.10. Let $-1 \leq x \leq 1$. Then $\sin(\sin^{-1}(x)) = 1$. Now, let $-\pi/2 \leq y \leq \pi/2$. Then $\sin^{-1}(\sin(y)) = y$, since the domain $-\pi/2 \leq y \leq \pi/2$ is where we defined sin to be one-to-one. However, note that for $y = \pi$, we have $\sin(\pi) = 0$, and $\sin^{-1}(0) = 0$, so

$$\sin^{-1}(\sin(\pi)) = \sin^{-1}(0) = 0 \neq \pi.$$

So, it is not always true that $\sin^{-1}(\sin(y)) = y$. We have to be careful with our domain.

Remark 5.11. To integrate $\sqrt{a^2 - x^2}$, we can use the substitution $x = a\sin\theta$. More generally, we can integrate $(a^2 - x^2)^{n/2}$ where n is any integer, using the substitution $x = a\sin\theta$

Remark 5.12. To integrate $\sqrt{a^2 + x^2}$, we can use the substitution $x = a\tan\theta$.

Remark 5.13. To integrate $\sqrt{x^2 - a^2}$, we use the substitution $x = a\sec\theta$.

5.4. **Partial Fractions.** This method of integration is perhaps best elucidated by an example.

Example 5.14. Suppose we want to integrate $\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)}$. It turns out that we can express this function as a sum of the form

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}.$$

To determine A and B, multiply both sides by $(x-1)(x+1)$ to get

$$1 = A(x+1) + B(x-1).$$

Plugging in $x = 1$, we get $1 = 2A$, so that $A = 1/2$. Plugging in $x = -1$, we get $1 = B(-2)$, so $B = -1/2$. In summary,

$$\frac{1}{(x-1)(x+1)} = \frac{1}{2(x-1)} - \frac{1}{2(x+1)}.$$

Therefore,

$$\int \frac{dx}{(x-1)(x+1)} = \int \frac{1}{2(x-1)}dx - \int \frac{1}{2(x+1)}dx = (1/2)\ln|x-1| - (1/2)\ln|x+1| + C.$$

More generally, if $a_1, \ldots, a_n$ are *distinct* real numbers, and if P is a polynomial of degree less than n, then we can find real numbers $A_1, \ldots, A_n$ such that

$$\frac{P(x)}{(x-a_1)\cdots(x-a_n)} = \frac{A_1}{x-a_1} + \cdots + \frac{A_n}{x-a_n}.$$

And this equality allows us to compute the integral of the left side.

In the case of repeated roots in the denominator, we need to add extra terms to the right side, as in the following example.

Example 5.15.

$$\frac{x}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2}$$

To find A, B, C we multiply both sides by $(x-1)(x+2)^2$ to get

$$x = A(x+2)^2 + B(x-1)(x+2) + C(x-1).$$

Letting $x = 1$, we get $1 = 9A$, so $A = 1/9$. Letting $x = -2$, we get $-2 = C(-3)$, so $C = 2/3$. So, we have

$$\begin{aligned} x &= (1/9)(x+2)^2 + B(x-1)(x+2) + (2/3)(x-1) \\ &= x^2(1/9 + B) + x(B + 4/9 + 2/3) + 4/9 - 2B - 2/3. \end{aligned}$$

So, we must have $B = -1/9$. In conclusion,

$$\frac{x}{(x-1)(x+2)^2} = \frac{1}{9(x-1)} - \frac{1}{9(x+2)} + \frac{2}{3(x+2)^2}.$$

Therefore,

$$\int \frac{x}{(x-1)(x+2)^2}dx = (1/9)\ln|x-1| - (1/9)\ln|x+2| - (2/3)(x+2)^{-1} + C.$$

Example 5.16. In the case that the degree of P is greater than or equal to the degree of the denominator, we first divide the numerator by the denominator using Euclid's division algorithm, as we now demonstrate. Suppose we want to integrate the function

$$\frac{x^3}{x^2-1}.$$

In general, we can perform long division on these polynomials using Euclid's division algorithm. In this case, we can just write

$$x^3 = x(x^2-1)+x.$$

Dividing both sides by x^2-1, we have

$$\frac{x^3}{x^2-1} = x + \frac{x}{x^2-1}.$$

The final term can now be treated using the method of partial fractions.

5.5. **Improper Integrals.** It is sometimes nice to integrate a function at infinity. We can do this via the following definition.

Definition 5.17. Fix a real number a. Suppose f is integrable on $[a,b]$ for all $b>a$. The **improper integral** of f on $[a,\infty)$ is defined as the following limit (if the limit exists):

$$\int_a^\infty f(x)dx = \lim_{R\to\infty}\int_a^R f(x)dx.$$

If this limit exists and is finite, we say that the improper integral **converges**. If the limit does not exist, we say that the improper integral **diverges**.

Remark 5.18. We similarly define

$$\int_{-\infty}^a f(x)dx = \lim_{R\to\infty}\int_{-R}^a f(x)dx.$$

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^0 f(x)dx + \int_0^\infty f(x)dx.$$

Example 5.19. Let $a>0$. Then

$$\int_0^\infty e^{-at}dt = \lim_{R\to\infty}\int_0^R e^{-at}dt = \lim_{R\to\infty}(-1/a)[e^{-aR}-1] = 1/a.$$

Theorem 5.20. *Let $a>0$. If $p>1$, then*

$$\int_a^\infty \frac{1}{x^p}dx = (p-1)^{-1}a^{-p+1}.$$

If $p\leq 1$, then $\int_a^\infty \frac{1}{x^p}dx$ diverges.

To see this, note that for $p\neq 1$ we have

$$\int_a^\infty \frac{1}{x^p}dx \lim_{R\to\infty}\int_a^R x^{-p}dx = \lim_{R\to\infty}(-p+1)^{-1}[R^{-p+1}-a^{-p+1}]$$

If $p>1$, then $\lim_{R\to\infty}R^{-p+1}=0$, so

$$\int_a^\infty \frac{1}{x^p}dx = (p-1)^{-1}a^{-p+1}.$$

If $p < 1$, then $\lim_{R\to\infty} R^{-p+1} = \infty$, so $\int_a^\infty \frac{1}{x^p} dx$ diverges.

In the remaining case $p = 1$, we have

$$\int_a^\infty \frac{1}{x} dx \lim_{R\to\infty} \int_a^R x^{-1} dx = \lim_{R\to\infty} (\ln |R| - \ln |a|).$$

Since $\lim_{R\to\infty} \ln |R| = \infty$, we conclude that $\int_a^\infty \frac{1}{x} dx$ diverges.

It is possible to similarly integrate the discontinuities of integrals, by approaching them in a limiting sense.

Definition 5.21. Let $a < b$. Suppose f is continuous on $[a, b)$ but discontinuous at b. We define the integral of f on $[a, b]$ as the following limit (if the limit exists):

$$\int_a^b f(x)dx = \lim_{R\to b^-} \int_a^R f(x)dx.$$

If this limit exists and is finite, we say that the improper integral **converges**. If the limit does not exist, we say that the improper integral **diverges**.

Remark 5.22. Similarly, if f is continuous on $(a, b]$ but discontinuous at a. We define the integral of f on $[a, b]$ as the following limit (if the limit exists):

$$\int_a^b f(x)dx = \lim_{R\to a^+} \int_R^b f(x)dx.$$

Remark 5.23 (**Integrating over a Discontinuity**)**.** If f is discontinuous only at c, if $a < c < b$, and if $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are finite, then we define

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Otherwise, we say that $\int_a^b f(x)dx$ diverges. For example, we the integral

$$\int_{-1}^1 \frac{dx}{x}$$

diverges, since $\int_0^1 dx/x$ diverges.

Example 5.24. Let $a > 0$. If $p < 1$, then

$$\int_0^a \frac{dx}{x^p} = (1 - p)^{-1} a^{1-p}.$$

If $p \geq 1$, then $\int_0^a dx/x^p$ diverges. To see this, let $p \neq 1$ and observe

$$\int_0^a \frac{dx}{x^p} = \lim_{R\to 0^+} \int_R^a \frac{dx}{x^p} = \lim_{R\to 0^+} (-p + 1)^{-1} [a^{-p+1} - R^{-p+1}].$$

If $p < 1$, then $\lim_{R\to 0^+} R^{-p+1} = 0$, so $\int_0^a \frac{dx}{x^p} = (1 - p)^{-1} a^{1-p}$. If $p > 1$, then $\lim_{R\to 0^+} R^{-p+1} = \infty$, so $\int_0^a \frac{dx}{x^p}$ diverges. In the remaining case $p = 1$, we have

$$\int_0^a \frac{dx}{x} = \lim_{R\to 0^+} \int_R^a \frac{dx}{x} = \lim_{R\to 0^+} [\ln |a| - \ln |R|] = -\infty.$$

So, the integral $\int_0^a \frac{dx}{x}$ diverges.

5.5.1. *Comparing Integrals.* We have reached somewhat of an end to our investigation of methods of integration. Sometimes, it is actually impossible to get an explicit formula of an integral. Instead, we need to know how to estimate integrals in various ways. In the next section, we will see a few ways that a computer can estimate an integral. For now, we will simply get some rough estimates on integrals. For example, we will just look at a way to check whether or not an improper integral converges.

Proposition 5.25. *Fix a real number a. Suppose $f(x) \geq g(x) \geq 0$ for all $x \geq a$.*

- *If $\int_a^\infty f(x)dx$ converges, then $\int_a^\infty g(x)dx$ converges.*
- *If $\int_a^\infty g(x)dx$ diverges, then $\int_a^\infty f(x)dx$ diverges.*

Example 5.26. We will demonstrate that $\int_1^\infty \frac{dx}{\sqrt{x^3+1}}$ converges.

Let $x \geq 1$. Then $\sqrt{x^3+1} \geq \sqrt{x^3} = x^{3/2} \geq 0$. So, $0 \leq (x^3+1)^{-1/2} \leq x^{-3/2}$. And

$$0 \leq \int_1^\infty x^{-3/2}dx = \lim_{R\to\infty} \int_1^R x^{-3/2}dx = \lim_{R\to\infty} (-2)(R^{-1/2}-1) = 2.$$

Therefore, $\int_1^\infty \frac{dx}{\sqrt{x^3+1}}$ converges.

We can do the same test for an endpoint discontinuity as follows.

Example 5.27. We will demonstrate that $\int_0^1 \frac{dx}{x^2+x^8}$ diverges. Let $0 < x \leq 1$. Note that $x^2+x^8 \leq 2x^2$, so $(x^2+x^8)^{-1} \geq (2x^2)^{-1} \geq 0$. However, we know that $\int_0^1 x^{-2}dx$ diverges. We therefore conclude that $\int_0^1 dx/(x^2+x^8)$ diverges.

5.6. **Numerical Integration.** As we discussed previously, some integrals simply cannot be evaluated with exact formulas. Thankfully, there are many ways that allow us to estimate these integrals. These methods are implemented in computers, allowing fairly precise estimation of various integrals. (However, be warned that arithmetic in computers has its own issues that need to be properly understood in order to make reliable estimates of integrals. For a hint of these issues, note that the typical computer says that $1+2^{-53}$ is equal to 1. And a calculator or phone app probably has worse precision than this. This imprecision of addition may seem innocuous, but if you start to subtract numbers that are close to one another, then e.g. $2^{-53}((1+2^{-53})-1)^{-1}$ should evaluate to 1, but it instead evaluates to an undefined number, since the computer attempts a division by zero.)

Example 5.28 (Trapezoidal Rule). Suppose we have $N+1$ equally spaced points on the interval $[a,b]$. We label these points as $a = x_0 < x_1 < \cdots < x_N = b$. Note that $x_i - x_{i-1} = (b-a)/N$ for each $1 \leq i \leq N$. We approximate the area under the curve of f by a set of trapezoids. Recall that a trapezoid of with w and heights h_1, h_2 has area $w(h_1+h_2)/2$. We will particularly approximate the area under f by the trapezoids of width $x_i - x_{i-1}$ and heights $f(x_i), f(x_{i-1})$, for each $1 \leq i \leq N$. The total area of all of these trapezoids is then

$$T_N = (x_1-x_0)\frac{f(x_1)+f(x_0)}{2} + (x_2-x_1)\frac{f(x_2)+f(x_1)}{2} + \cdots + (x_N-x_{N-1})\frac{f(x_N)+f(x_{N-1})}{2}.$$

Using $x_i - x_{i-1} = (b-a)/N$ for each $1 \leq i \leq N$, the total area of the trapezoids is equal to

$$T_N = \frac{b-a}{N}\left(f(x_0)/2 + f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_{N-1}) + f(x_N)/2\right).$$

Remark 5.29. Another way of describing the trapezoidal rule is that we have approximated our function f by a piecewise linear function g, and we then use the integral of g to approximate the integral of f.

Remark 5.30 (Midpoint Rule). If is also possible to approximate a function f by its Riemann sums (as in the midpoint rule). In particular, the Midpoint Rule approximates a function f by a Riemann sum of equally spaced points, where we evaluate the function f at the midpoint of each rectangle. That is, we approximate the integral of f by

$$M_N = \frac{b-a}{N}\left(f\left(\frac{x_0+x_1}{2}\right)+f\left(\frac{x_1+x_2}{2}\right)+\cdots+f\left(\frac{x_{N-1}+x_N}{2}\right)\right).$$

Another way of describing the Midpoint Rule is that we approximate f by a piecewise constant function g, and we then use the integral of g to approximate the integral of f.

Theorem 5.31 (Error Bounds for Trapezoid and Midpoint Rules). *Let f be an integrable function on $[a,b]$. Assume that f'' exists and is continuous. Suppose $|f''(x)| \leq K$ for all $a \leq x \leq b$. Then*

$$\left|\int_a^b f(x)dx - T_N\right| \leq \frac{K(b-a)^3}{12N^2}.$$

$$\left|\int_a^b f(x)dx - M_N\right| \leq \frac{K(b-a)^3}{24N^2}.$$

Example 5.32. Let's compute both the trapezoid and midpoint rules for a function, and verify that these error rates are correct. We will estimate $\int_1^2 \sqrt{x}dx$ with $N=6$. Since $b=2$ and $a=1$, we have $(b-a)/N = 1/6$. The points $x_0,\ldots,x_N$ are $1, 7/6, 8/6, 9/6, 10/6, 11/6, 2$. And

$$T_6 = \frac{1}{6}\left(\sqrt{1}/2+\sqrt{7/6}+\sqrt{4/3}+\sqrt{3/2}+\sqrt{5/3}+\sqrt{11/6}+\sqrt{2}/2\right) \approx 1.218612.$$

$$M_6 = \frac{1}{6}\left(\sqrt{13/12}+\sqrt{15/12}+\sqrt{17/12}+\sqrt{19/12}+\sqrt{21/12}+\sqrt{23/12}\right) \approx 1.219121.$$

In this case, we can compute the integral exactly:

$$\int_1^2 \sqrt{x}dx = (2/3)(2^{3/2}-1^{3/2}) = (2/3)(2^{3/2}-1) \approx 1.218951.$$

For $f(x)=\sqrt{x}$, we have $f'(x)=(1/2)x^{-1/2}$ and $f''(x)=(-1/4)x^{-3/2}$, so for $1 \leq x \leq 2$, we have $|f''(x)| \leq 1/4$. So, we can verify our Theorem for Error Bounds as follows

$$.000339 \approx \left|\int_1^2 \sqrt{x}dx - T_6\right| \leq \frac{K(b-a)^3}{12N^2} = \frac{(1/4)(1)}{12(36)} = \frac{1}{1728} \approx .000579.$$

$$.000170 \approx \left|\int_1^2 \sqrt{x}dx - M_6\right| \leq \frac{K(b-a)^3}{24N^2} = \frac{(1/4)(1)}{24(36)} = \frac{1}{3456} \approx .000289.$$

So, in this case, our error bound is actually not too much larger than the actual error between the integral and its approximations.

The following problem is fairly typical when we try to evaluate an integral with a computer. We want to approximate a certain integral, and we want to guarantee that our approximation is a certain distance from the correct answer.

Example 5.33. Find an integer N such that we can guarantee that T_N approximates $\int_0^3 e^{-x^2}dx$ within an error of 10^{-5}.

We first estimate the second derivative of $f(x) = e^{-x^2}$. Then $f'(x) = (-2x)e^{-x^2}$ and $f''(x) = (4x^2-2)e^{-x^2}$. Let $0 \leq x \leq 3$. Then $|4x^2-2| \leq 34$ and $|e^{-x^2}| \leq 1$, so $|f''(x)| \leq 34$ for $0 \leq x \leq 3$. Using the error bound for T_N, we want to find N such that

$$\frac{34(b-a)^3}{12N^2} < 10^{-5}.$$

Since $(b-a) = 3$, we want to find N such that

$$N > \sqrt{\frac{34(3^3)(10^5)}{12}} \approx 2765.9$$

Therefore, choosing $N = 2766$ suffices.

So far, we have seen the Midpoint Rule, which approximates f by a piecewise constant function g and then computes the integral of g as an approximation to the integral of f. We also saw the Trapezoid Rule, which approximates f by a piecewise linear function g and then computes the integral of g as an approximation to the integral of f. We now take this idea one step further. With **Simpson's Rule**, we approximate f by a piecewise quadratic function g and then compute the integral of g as an approximation to the integral of f. After some analysis (which we omit), Simpson's rule S_N for an even N has the following formula

$$S_N = \frac{1}{3}T_{N/2} + \frac{2}{3}M_{N/2}.$$

Substituting the formulas for $T_{N/2}$ and $M_{N/2}$ into this formula, we get

$$S_N = \frac{(b-a)}{3N}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{N-2}) + 4f(x_{N-1}) + f(x_N)\right).$$

We then get the following error bound for Simpson's Rule

Theorem 5.34 (Error Bound for Simpson's Rule). *Let f be an integrable function on $[a,b]$. Assume that $f^{(4)}$ exists and is continuous. Suppose $\left|f^{(4)}x)\right| \leq K$ for all $a \leq x \leq b$. Then*

$$\left|\int_a^b f(x)dx - S_N\right| \leq \frac{K(b-a)^5}{180N^4}.$$

Example 5.35. We continue our above example, and this time we use Simpson's rule. We estimate $\int_1^2 \sqrt{x}dx$ with $N = 6$. Since $b = 2$ and $a = 1$, we have $(b-a)/N = 1/6$. The points $x_0, \ldots, x_N$ are $1, 7/6, 8/6, 9/6, 10/6, 11/6, 2$. And

$$S_6 = \frac{1}{18}\left(\sqrt{1} + 4\sqrt{7/6} + 2\sqrt{4/3} + 4\sqrt{3/2} + 2\sqrt{5/3} + 4\sqrt{11/6} + \sqrt{2}\right) \approx 1.21895013.$$

As before, we can compute the integral exactly

$$\int_1^2 \sqrt{x}dx = (2/3)(2^{3/2} - 1^{3/2}) = (2/3)(2^{3/2} - 1) \approx 1.21895142.$$

For $f(x) = \sqrt{x}$, we have $f'(x) = (1/2)x^{-1/2}$, $f''(x) = (-1/4)x^{-3/2}$, $f'''(x) = (3/8)x^{-5/2}$, and $f^{(4)}(x) = -(15/16)x^{-7/2}$. So for $1 \leq x \leq 2$, we have $\left|f^{(4)}(x)\right| \leq 15/16$. So, we can verify our

Theorem for Error Bounds as follows

$$.00000129 \approx \left| \int_1^2 \sqrt{x}dx - S_6 \right| \leq \frac{K(b-a)^5}{180N^4} = \frac{(15/16)(1)}{180(6^4)} = \frac{1}{248832} \approx .00000402.$$

Note that in this case, Simpson's rule is roughly 100 times more accurate than the Midpoint or Trapezoid rules, even though we used the same number of sample points.

6. More Applications of Integrals

6.1. **Arc Length.** Consider a linear function on the interval $[a,b]$. That is, $y = cx + d$. What is the length of this curve on the interval $[a,b]$? Consider the triangle with vertices $(a, y(a))$, $(b, y(b))$ and $(b, y(a))$. The base of this right triangle has length $(b-a)$, and its height is $y(b) - y(a) = c(b-a)$. So, it's hypotenuse has length

$$\sqrt{(b-a)^2 + c^2(b-a)^2} = (b-a)\sqrt{1+c^2} = (b-a)\sqrt{1+(y'(x))^2}.$$

In conclusion, the function $y = cx + d$ has length $\sqrt{1 + y'(x)^2}\,(b-a)$ on the interval $[a,b]$. Since a continuously differentiable function is locally piecewise linear, we therefore define the arc length of a general function $f(x)$ on the interval $[a,b]$ by

$$\int_a^b \sqrt{1 + f'(x)^2}dx.$$

Example 6.1. Consider the function $f(x) = x^2/2$. The arc length of this function from $x = 0$ to $x = 1$ is computed using the substitution $x = \tan\theta$ as follows.

$$\begin{aligned}
\int_0^1 \sqrt{1+f'(x)^2}dx &= \int_0^1 \sqrt{1+x^2}dx = \int_0^{\pi/4} \sqrt{1+\tan^2\theta}(\cos\theta)^{-2}d\theta \\
&= \int_0^{\pi/4} \sqrt{1+\tan^2\theta}(\cos\theta)^{-2}d\theta = \int_0^{\pi/4} (\cos\theta)^{-3}d\theta \\
&= \int_0^{\pi/4} (\cos\theta)^{-1}(d/d\theta)\tan\theta d\theta \\
&= [(\cos\theta)^{-1}\tan\theta]_{\theta=0}^{\theta=\pi/4} - \int_0^{\pi/4} \tan\theta(-1)(\cos\theta)^{-2}(-\sin\theta)d\theta \\
&= 2/\sqrt{2} - \int_0^{\pi/4} \sin^2\theta(\cos\theta)^{-3}d\theta \\
&= 2/\sqrt{2} - \int_0^{\pi/4} (\cos\theta)^{-3}d\theta + \int_0^{\pi/4} (\cos\theta)^{-1}d\theta.
\end{aligned}$$

So, using $\int(\cos\theta)^{-1}d\theta = \ln|\sec\theta + \tan\theta|$,

$$\begin{aligned}
\int_0^1 \sqrt{1+f'(x)^2}dx &= 1/\sqrt{2} + (1/2)[\ln|\sec\theta + \tan\theta|]_{\theta=0}^{\theta=\pi/4} \\
&= 1/\sqrt{2} + (1/2)[\ln(2/\sqrt{2}+1)] \approx 1.1478.
\end{aligned}$$

As a check, note that the straight line between the points $(0,0)$ and $(1,1/2)$ has length $\sqrt{1+1/4} \approx 1.118$, and we expect this shortest path to have slightly greater length than the curved path $f(x) = x^2/2$.

6.1.1. *Surface Area of Revolution.* Suppose we have a curve $y = f(x)$, and we then rotate this curve around the x-axis, producing a surface in three-dimensional Euclidean space. A short segment of this curve near x has length approximated by the integral of $\sqrt{1 + f'(x)^2}$, as we discussed above. If we rotate this segment of the curve around the x-axis, then the resulting surface is approximately a cylinder, so it has area $2\pi r\sqrt{1 + f'(x)^2}$, where $r = f(x)$ is the approximate radius of the cylinder. That is, near x, the rotated curve has approximate area $2\pi f(x)\sqrt{1 + f'(x)^2}$. We therefore define the surface area of the curve $y = f(x)$ for $a \leq x \leq b$ rotated around the x-axis by

$$2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\, dx.$$

Example 6.2. We compute the area of the sphere of radius R. This sphere is obtained by rotating the curve $f(x) = \sqrt{R^2 - x^2}$ on the interval $[-R, R]$ around the x-axis. So, the sphere of radius R has area

$$2\pi \int_{-R}^R \sqrt{R^2 - x^2}\sqrt{1 + \frac{x^2}{R^2 - x^2}}dx = 2\pi \int_{-R}^R \sqrt{R^2 - x^2}\sqrt{\frac{R^2}{R^2 - x^2}}dx = 2\pi \int_{-R}^R R dx = 4\pi R^2.$$

6.2. **Fluid pressure and Force.**

Definition 6.3 (**Fluid Pressure**)**.** The pressure p at depth h in a fluid of mass density ρ is

$$p = \rho g h.$$

For example, h is in meters, g is the gravitational constant (approximately 9.8 meters per second^2), and ρ can be measured in kilograms per meter^3.

When we are under water, we can think of the pressure of a fluid like a column of water that is over us. Similarly, we can think of atmospheric pressure as a result of a tall column of air that is sitting on our heads. However, this intuition is a bit deceiving, since pressure does not act in any specific direction. Pressure is exerted on all parts of a surface. We feel the pressure of air not just on the top of our heads.

Definition 6.4 (**Fluid Force, for Constant Pressure**)**.** Suppose we have a surface of area A. If the pressure p on S is constant, then the force of a fluid on that surface is given by

$$pA.$$

Example 6.5. Suppose we have a flat square of cardboard measuring 4 meters by 4 meters at a depth of 3 meters in water. We use the density of water as $\rho \approx 10^3$ kg/m^3. Then the force on the cardboard is

$$\rho g h A = 10^3(9.8)(3)(4^2) = 470400\ N \qquad (N = kg \cdot m/s^2)$$

Similarly, if a different square of cardboard is at a depth of 6 meters, then there is twice as much force on the cardboard, namely

$$940800\ N$$

Now, suppose these two squares are connected by a single vertical surface. This edge has width 4 meters and height 3 meters. A small slice of this cardboard of thickness ε at height h has a force approximately

$$\rho g h 4\varepsilon.$$

So, the whole edge of the rectangle has a force which is given by the following integral

$$\rho g \int_{h=3}^{h=6} 4h dh = \rho g(4)(1/2)(36-9).$$

More generally, we have the following force calculation

Proposition 6.6 (**Fluid Force, Vertically Oriented Surface**). *Suppose we have a flat surface S sitting in a fluid, where S is oriented vertically, starting at height a and ending at height b, where $a < b$. Suppose also that this surface has width $f(y)$ at height y. Then the force exerted on this surface is given by*

$$\rho g \int_a^b y f(y) dy.$$

Example 6.7. We compute the fluid force of water on an equilateral triangle of side length 2 meters submerged vertically, so the top vertex of the triangle just touches the surface of the water. The force is then given by

$$\rho g \int_0^{\sqrt{3}} y(2y/\sqrt{3}) dy = 10^3(9.8)(2/\sqrt{3}) \int_0^{\sqrt{3}} y^2 dy = 10^3(9.8)(2/\sqrt{3})(1/3)3^{3/2} = 19600\ N$$

6.3. **Taylor polynomials, Taylor's Theorem.** We are now going to discuss a powerful tool for approximating functions with many derivatives. In essence, we will take a general function, and look at it in a very simple way.

Recall that a continuously differentiable function f can be approximated at $x = a$ by the following function of x:

$$L(x) = f(a) + (x-a)f'(a).$$

This approximation is known as the **first order** approximation of f at a.

Suppose now that all higher derivatives of f exist on some interval I. Let $a \in I$. We define the **nth order Taylor polynomial** at $x = a$ as follows

$$T_n(x) = f(a) + (x-a)f'(a) + (x-a)^2\frac{f''(a)}{2!} + (x-a)^3\frac{f'''(a)}{3!} + \cdots + (x-a)^n\frac{f^{(n)}(a)}{n!}.$$

Note that T_n is an n^{th} degree polynomial. For example,

$$T_1(x) = f(a) + (x-a)f'(a)$$
$$T_2(x) = f(a) + (x-a)f'(a) + (x-a)^2 f''(a)/2$$
$$T_3(x) = f(a) + (x-a)f'(a) + (x-a)^2 f''(a)/2 + (x-a)^3 f'''(a)/6.$$

Note that $T_1(x) = L(x)$. Also, note that f and T_n **agree to order n** at $x = a$. That is,

$$f(a) = T_n(a), \quad f'(a) = T_n'(a), \quad f''(a) = T_n''(a), \quad \cdots \quad f^{(n)}(a) = T_n^{(n)}(a).$$

Also,

$$T_n(x) = T_{n-1}(x) + (x-a)^n\frac{f^{(n)}(a)}{n!}.$$

If we use the convention $0! = 1! = 1$, and $f^{(0)}(a) = f(a)$, we can write T_n in the following compact way.

$$T_n(x) = \sum_{j=0}^{n} (x-a)^j \frac{f^{(j)}(a)}{j!}.$$

When $a = 0$, $T_n(x)$ is called the *nth* **MacLaurin polynomial** of f.

So far, we have not really discussed integrals at all, but they will enter the picture when we discuss how closely T_n approximates f.

Example 6.8. Consider the function $f(x) = e^x$. We will compute the Maclaurin polynomial of f, i.e. the Taylor polynomial of f at $x = 0$. Since $f^{(n)}(0) = e^0 = 1$ for all positive integers n, we have

$$T_1(x) = 1 + x$$
$$T_2(x) = 1 + x + x^2/2$$
$$T_3(x) = 1 + x + x^2/2 + x^3/6$$

And in general,

$$T_n(x) = \sum_{j=0}^{n} \frac{x^j}{j!}.$$

In fact, in a sense that we will describe in the next chapter, the exponential function is equal to the following infinite sum

$$e^x = \lim_{n\to\infty} \sum_{j=0}^{n} \frac{x^j}{j!}.$$

However, this type of identity is **not** true for a general function.

Example 6.9. Consider the function $f(x) = -\ln(1-x)$. We will compute the Maclaurin polynomial of f, i.e. the Taylor polynomial of f at $x = 0$. Note that $f(0) = 0$, $f'(x) = 1/(1-x)$, $f''(x) = 1/(1-x)^2$, $f'''(x) = 2/(1-x)^3$, and in general,

$$f^{(n)}(x) = \frac{(n-1)!}{(1-x)^n}. \qquad (*)$$

Indeed, assuming this formula holds for a given n by induction, we compute

$$f^{(n+1)}(x) = (f^{(n)}(x))' = \frac{-(n-1)!(-n)(1-x)^{n-1}}{(1-x)^{2n}} = \frac{n!}{(1-x)^{n+1}}.$$

And since $f'(x) = 1/(1-x) = 0!/(1-x)$, we conclude that $(*)$ holds for all positive integers n. So,

$$f^{(n)}(0) = (n-1)!$$

And

$$T_1(x) = x$$
$$T_2(x) = x + x^2/2$$
$$T_3(x) = x + x^2/2 + x^3/3$$

And in general,

$$T_n(x) = \sum_{j=1}^{n} \frac{x^j (j-1)!}{j!} = \sum_{j=1}^{n} \frac{x^j}{j}.$$

The meaning of the this polynomial as $n \to \infty$ now becomes more subtle than before. As long as $|x| < 1$, we have

$$-\ln(1-x) = \lim_{n\to\infty} \sum_{j=0}^{n} \frac{x^j}{j}.$$

However, this identity has no decipherable meaning when $|x| \geq 1$. However, this is expected, since $-\ln(1-x)$ is undefined at $x = 1$. We will discuss these topics more in the following chapter.

Example 6.10. Consider the function $f(x) = \sin(x)$. We will compute the Maclaurin polynomial of f, i.e. the Taylor polynomial of f at $x = 0$. Note that $f'(x) = \cos(x)$, $f''(x) = -\sin(x)$, $f'''(x) = -\cos(x)$ and $f^{(4)}(x) = \sin(x)$. That is, the sequences of numbers $f^{(0)}(0)$, $f^{(1)}(0)$, $f^{(2)}(0), \ldots$ is a repeating sequence of the form

$$0, 1, 0, -1, 0, 1, 0, -1, 0, 1, 0, -1, \ldots$$

So,

$$\begin{aligned} T_1(x) &= x \\ T_2(x) &= x \\ T_3(x) &= x - x^3/6 \\ T_5(x) &= x - x^3/3! + x^5/5! \end{aligned}$$

And in general, when n is a nonnegative integer,

$$T_{2n+1}(x) = \sum_{j=1}^{n} \frac{(-1)^j}{(2j+1)!} x^{2j+1}, \qquad T_{2n+2}(x) = T_{2n+1}(x).$$

Similar to the case of the exponential function, the following expression holds for all x.

$$\sin(x) = \lim_{n \to \infty} \sum_{j=1}^{n} \frac{(-1)^j}{(2j+1)!} x^{2j+1}.$$

Example 6.11. Consider the function $f(x) = \cos(x)$. We will compute the Maclaurin polynomial of f, i.e. the Taylor polynomial of f at $x = 0$. Note that $f'(x) = -\sin(x)$, $f''(x) = -\cos(x)$, $f'''(x) = \sin(x)$ and $f^{(4)}(x) = \cos(x)$. That is, the sequences of numbers $f^{(0)}(0)$, $f^{(1)}(0)$, $f^{(2)}(0), \ldots$ is a repeating sequence of the form

$$1, 0, -1, 0, 1, 0, -1, 0, 1, 0, -1, 0, \ldots$$

So,

$$\begin{aligned} T_1(x) &= 1 \\ T_2(x) &= 1 - x^2/2 \\ T_3(x) &= 1 - x^2/2 \\ T_4(x) &= 1 - x^2/2 + x^4/24 \end{aligned}$$

And in general, when n is a positive integer,

$$T_{2n}(x) = \sum_{j=0}^{n} \frac{(-1)^j}{(2j)!} x^{2j}, \qquad T_{2n+1}(x) = T_{2n}(x).$$

Similar to the case of the exponential function and cos, the following expression holds for all x.

$$\cos(x) = \lim_{n \to \infty} \sum_{j=0}^{n} \frac{(-1)^j}{(2j)!} x^{2j}.$$

Remark 6.12. When your calculator or computer or other electronic device computes the value of cosine or sine or some other function that does not have an explicit algebraic formula, the electronic device is essentially computing a the value of a Taylor polynomial. For example, the standard way of computing $\sin(x)$ on a computer for $x \in [-\pi/2, \pi/2]$ is to instead calculate $T_{19}(x)$, the 19^{th} degree Maclaurin polynomial of sine. This is a sensible thing to do, since the error between T_{19} and sin is very small, as we now discuss.

Theorem 6.13 (Taylor's Theorem, Integral Remainder). *Let f be a real valued function, let n be a positive integer, and let a be a real number. Suppose f has Taylor polynomial T_n at $x = a$. Assume that $f^{(n+1)}$ exists and is continuous. Then*

$$f(x) = T_n(x) + \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)dt.$$

To prove that this formula holds, we use integration by parts to write

$$\begin{aligned}
&\frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)dt = \frac{1}{n!}\int_a^x (x-t)^n (d/dt) f^{(n)}(t)dt \\
&= -\frac{1}{n!}(x-a)^n f^{(n)}(a) - \frac{1}{n!}\int_a^x (d/dt)(x-t)^n f^{(n)}(t)dt \\
&= -\frac{1}{n!}(x-a)^n f^{(n)}(a) + \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f^{(n)}(t)dt \\
&= -\frac{1}{n!}(x-a)^n f^{(n)}(a) + \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} \frac{d}{dt} f^{(n-1)}(t)dt \\
&= -\frac{1}{n!}(x-a)^n f^{(n)}(a) - \frac{1}{(n-1)!}(x-a)^{n-1} f^{(n-1)}(a) - \frac{1}{(n-1)!}\int_a^x \frac{d}{dt}(x-t)^{n-1} f^{(n-1)}(t)dt \\
&= -\frac{1}{n!}(x-a)^n f^{(n)}(a) - \frac{1}{(n-1)!}(x-a)^{n-1} f^{(n-1)}(a) + \frac{1}{(n-2)!}\int_a^x (x-t)^{n-2} f^{(n-1)}(t)dt \\
&= \cdots .
\end{aligned}$$

We continue in this way, arriving at

$$\begin{aligned}
&\frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)dt \\
&= -\frac{1}{n!}(x-a)^n f^{(n)}(a) - \frac{1}{(n-1)!}(x-a)^{n-1} f^{(n-1)}(a) - \cdots - \frac{1}{1!}(x-a)^1 f'(a) + \int_a^x f'(t)dt \\
&= -\frac{1}{n!}(x-a)^n f^{(n)}(a) - \frac{1}{(n-1)!}(x-a)^{n-1} f^{(n-1)}(a) - \cdots - \frac{1}{1!}(x-a) f'(a) - f(a) + f(x) \\
&= -T_n(x) + f(x).
\end{aligned}$$

where we used the Fundamental Theorem of Calculus in the penultimate step.

There are a few different ways that we can use this integral remainder term. Here is one way to make an estimate of the error of the Taylor series.

Proposition 6.14 (An Error Bound for Taylor's Theorem). *Let f be a real valued function, let n be a positive integer, and let a be a real number. Suppose f has Taylor polynomial T_n at $x = a$. Assume that $f^{(n+1)}$ exists and is continuous. Let $K > 0$ such that*

$\left|f^{(n+1)}(t)\right| \leq K$ *for all* t *between* x *and* a. *Then*

$$|f(x) - T_n(x)| \leq K\frac{|x-a|^{n+1}}{(n+1)!}.$$

To prove this bound, we use Taylor's Theorem to get

$$\begin{aligned}|f(x) - T_n(x)| &= \left|\frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)dt\right| \leq \frac{1}{n!}\left|\int_a^x |x-t|^n \left|f^{(n+1)}(t)\right| dt\right| \\ &\leq \frac{K}{n!}\left|\int_a^x |x-t|^n\, dt\right| = \frac{K}{n!}\left|\int_a^x (x-t)^n dt\right| \\ &= \frac{K}{n!}\frac{1}{n+1}\left|(-1)(x-t)^{n+1}|_{t=a}^{t=x}\right| = \frac{K}{(n+1)!}|x-a|^{n+1}.\end{aligned}$$

Example 6.15. Let's see that the 20^{th} order Taylor expansion of $f(x) = \sin(x)$ is very accurate near $x = 0$, so that we can trust our computers when they compute this function. Let $T_n(x)$ be the Taylor expansion of f at $x = 0$, and then let x be a variable point in $[-\pi/2, \pi/2]$. Since $\left|f^{(n)}(x)\right| \leq 1$ for all x and for all positive integers n, we have

$$|f(x) - T_{20}(x)| \leq \frac{1}{(21!)}|x|^{21} \leq \frac{1}{21!}(\pi/2)^{21} \approx 2.5 \times 10^{-16}.$$

Since the decimals precision of a typical computer is around 2.22×10^{-16}, we see that this Taylor polynomial was chosen to be just as accurate as any other computation. In other words, we can pretty reliably trust this Taylor expansion.

7. Sequences and Infinite Series

7.1. **Introduction.** As we saw in the previous section, many functions can be well approximated by their nth degree Taylor polynomials, by taking n to be large. Perhaps surprisingly, some functions such as the exponential, sine and cosine functions are *equal to* their Taylor polynomials, if we let n go to infinity. In other words, we can begin with a presumably complicated function and break it into an infinite number of simpler pieces. Specifically, note that monomials are some of the simplest possible functions to encounter. In some sense, we can therefore consider a Taylor expansion (as $n \to \infty$) as breaking up functions into an infinite number of constituent elements or atoms.

As we also discussed in the previous section, we need to be careful when dealing with infinite sums of monomials. As we briefly mentioned in the case of $-\ln(1-x)$, its Taylor polynomial only makes sense for $|x| < 1$, when we let $n \to \infty$. We will therefore devote some time to making careful statements about the convergence and divergence of infinite sums. Afterwards, we will be able to discuss infinite sum representations of functions such as the exponential, sine and cosine.

7.2. **Sequences.**

Definition 7.1. A **sequence** is a real valued function f where the domain of f is a set of integers. The values $a_n = f(n)$ are the **terms** of the sequence, where n is an integer which is referred to as the **index** of the sequence. Informally, a sequence is a list of numbers $a_1, a_2, a_3, \ldots$. We denote the sequence of numbers by $\{a_n\}$.

Example 7.2. For each positive integer n, let $a_n = 1/n$. Then $a_1 = 1$, $a_2 = 1/2$, $a_3 = 1/3$, and so on.

Example 7.3. For each positive integer n, let $a_n = (-1)^n$. Then $a_1 = -1$, $a_2 = 1$, $a_3 = -1$, and so on.

Example 7.4 (The Babylonian Square Root Algorithm). The following recursively defined sequence gets closer and closer to the square root of 2.

Define $a_1 = 1$, and for any integer $n \geq 2$,

$$a_n = \frac{1}{2}\left(a_{n-1} + \frac{2}{a_{n-1}}\right).$$

We compute

$$a_1 = \frac{1}{2}(1+2) = 1.5.$$

$$a_2 = \frac{1}{2}(3/2 + 4/3) = 17/12 \approx 1.4167$$

$$a_3 = \frac{1}{2}(17/12 + 24/17) = 577/408 \approx 1.414216$$

Indeed, it can be shown that a_n gets arbitrarily close to $\sqrt{2}$ as n tends to infinity. We will make this statement more precise below.

Note that if we defined $a_n = \frac{1}{2}\left(a_{n-1} + \frac{M}{a_{n-1}}\right)$, then this sequence will get closer and closer to $\sqrt{M}$, where $M \geq 0$. This phenomenon can be explained by **Newton's Method**. We have

$$a_n = a_{n-1} - (1/2)a_{n-1} - \frac{M}{2a_{n-1}} = a_{n-1} - \frac{(a_{n-1}^2 - M)}{2a_{n-1}} = a_{n-1} - \frac{f(a_{n-1})}{f'(a_{n-1})},$$

where $f(x) = x^2 - M$. This iterative scheme searches for a zero of the function f. Given the x-value a_{n-1}, the linear approximation of f at a_{n-1} is $f'(a_{n-1})(x - a_{n-1}) + f(a_{n-1})$. And we define a_n as the x-intercept of this line. That is, we define a_n so that

$$0 = f'(a_{n-1})(a_n - a_{n-1}) + f(a_{n-1}).$$

Solving this equation for a_n gives the above equality.

$$a_n = a_{n-1} - \frac{f(a_{n-1})}{f'(a_{n-1})}.$$

Remark 7.5. The Babylonian square root algorithm is typically used by computers when you ask for the square root of a number. That is, the computer will compute something like a_{15} for a given number M, and then return a_{15} as the square root of the number M.

Definition 7.6 (Limit of a Sequence). Let $\{a_n\}$ be a sequence and let L be a real number. We say that $\{a_n\}$ **converges** to L if and only if, for every $\varepsilon > 0$, there exists an integer $M = M(\varepsilon)$ such that $|a_n - L| < \varepsilon$ for all $n > M$. If $\{a_n\}$ converges to L, we write

$$\lim_{n\to\infty} a_n = L, \qquad \text{or} \qquad a_n \to L.$$

If no such limit L exists, we say that $\{a_n\}$ **diverges**. If, for any $K > 0$, there exists $M = M(K)$ such that $a_n > K$ for all $n > M$, we say that $\{a_n\}$ **diverges to infinity**.

Remark 7.7. If a sequence $\{a_n\}$ converges to some real number L, then L is the unique real number such that $\{a_n\}$ converges to L.

Example 7.8. Let's prove that the sequence $a_n = 1/n$ converges to 0 as $n \to \infty$. Note that 0 is never a member of the sequence, but the sequence can still converge to zero.

Let $\varepsilon > 0$. We first find $M = M(\varepsilon)$ such that $1/n < \varepsilon$ for all $n > M$. Let M be an integer such that $M > 1/\varepsilon$. Note that M is positive and $1/M < \varepsilon$. If $n > M$, then $1/n < 1/M < \varepsilon$. So, for all $n > M$, we have

$$|a_n - 0| = |1/n| = 1/n < \varepsilon.$$

We conclude that $0 = \lim_{n\to\infty} a_n$.

Example 7.9. The sequence $a_n = (-1)^n$ diverges.

Example 7.10. The sequence $a_n = n$ diverges to infinity.

Remark 7.11. Suppose a sequence $\{a_n\}$ converges to some real number L. Then $\{a_n\}$ still converges to L if we change any finite number of terms of the sequence $\{a_n\}$.

Remark 7.12. Suppose C is a constant, and there is an integer M such that $a_n = C$ for all $n > M$. Then $\lim_{n\to\infty} a_n = C$.

Proposition 7.13. *Let f be a function on the real line, and define $a_n = f(n)$ for every positive integer n. If $\lim_{x\to\infty} f(x)$ exists, then $\lim_{n\to\infty} a_n$ exists as well, and*

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x).$$

Example 7.14. This proposition allows us to use L'Hopital's Rule as follows.

$$\lim_{x\to\infty} \frac{\ln x}{x^2} = \lim_{x\to\infty} \frac{x^{-1}}{2x} = 0.$$

Therefore,

$$\lim_{n\to\infty} \frac{\ln n}{n^2} = 0.$$

We can restate several theorems about limits of functions to theorems about limits of sequences.

Proposition 7.15 (Limit Laws for Sequences). *Let $\{a_n\}, \{b_n\}$ be a sequences and let L, M be real numbers. Assume that $a_n \to L$ and $b_n \to M$. (This assumption is **very important** in what follows.) Then*

- $\lim_{n\to\infty}(a_n + b_n) = (\lim_{n\to\infty} a_n) + (\lim_{n\to\infty} b_n) = L + M$.
- $\lim_{n\to\infty}(a_n - b_n) = (\lim_{n\to\infty} a_n) - (\lim_{n\to\infty} b_n) = L - M$.
- $\lim_{n\to\infty}(a_n b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n) = LM$.
- *If $M \neq 0$, then* $\lim_{n\to\infty}(a_n/b_n) = (\lim_{n\to\infty} a_n)/(\lim_{n\to\infty} b_n) = L/M$.
- $\lim_{n\to\infty}(c a_n) = c(\lim_{n\to\infty} a_n) = cL$.

Example 7.16. The sequences $a_n = (-1)^n$ and $b_n = -(-1)^n$ both diverge, but $a_n + b_n = 0$, so $a_n + b_n \to 0$. So, the assumptions in this proposition are really needed.

Theorem 7.17 (Squeeze Theorem for Sequences). *Let $\{a_n\}, \{b_n\}, \{c_n\}$ be a sequences and let L be a real number. Assume that*

$$\lim_{n\to\infty} a_n = L, \quad \textit{and} \quad \lim_{n\to\infty} c_n = L.$$

Assume also that there exists an integer M such that, for all $n > M$,

$$a_n \leq b_n \leq c_n.$$

Then $\lim_{n\to\infty} b_n = L$.

Example 7.18. Let R be a real number. We will show that

$$\lim_{n\to\infty} \frac{R^n}{n!} = 0.$$

First, suppose R is any integer. Once we have $n > R$, then we write

$$\frac{R^n}{n!} = \frac{R}{1}\frac{R}{2}\cdots\frac{R}{R}\frac{R}{R+1}\frac{R}{R+2}\cdots\frac{R}{n}.$$

Write $C = (R/1)(R/2)\cdots(R/R)$, and note that the remaining terms in the product $R^n/n!$ are less than 1 in absolute value. So, ignoring all other terms except for the last one,

$$\left|\frac{R^n}{n!}\right| \leq |C|\frac{|R|}{n}.$$

So, define $a_n = -|CR|/n$, and define $c_n = |CR|/n$. Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = 0$ and $a_n \leq R^n/n! \leq c_n$. So, the sequence $b_n = R^n/n!$ satisfies $\lim_{n\to\infty} b_n = 0$, by the Squeeze Theorem, as desired.

The following Theorem says that a continuous function commutes with limits of sequences.

Theorem 7.19. *Suppose f is a continuous function, $\{a_n\}$ is a sequence, and $a_n \to L$ for some real number L. Then*

$$\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(L).$$

Example 7.20. We have already used this Theorem implicitly before. For example, to compute $\lim_{x\to 0^+} x^x$, we let a_n be any positive sequence such that $a_n \to 0$. Then

$$\lim_{x\to 0+} x^x = \lim_{n\to\infty} a_n^{a_n} = \lim_{n\to\infty} e^{a_n \ln a_n} = e^{\lim_{n\to\infty} a_n \ln a_n} = e^0 = 1.$$

That is, we used the fact that the exponential function is continuous.

Example 7.21.

$$\lim_{n\to\infty} \sin\left(\frac{1}{n+1}\right) = \sin\left(\lim_{n\to\infty}\frac{1}{n+1}\right) = \sin(0) = 0.$$

The following proposition is sometimes useful.

Proposition 7.22 (Bounded Monotonic Sequences Converge). *Let $\{a_n\}$ be a strictly increasing sequence. That is, $a_{n+1} > a_n$ for all positive integers n. Suppose also there exists a real number M such that $a_n < M$ for all positive integers n. Then $\{a_n\}$ converges, and* $\lim_{n\to\infty} a_n \leq M$.

Let $\{a_n\}$ be a strictly decreasing sequence. That is, $a_{n+1} < a_n$ for all positive integers n. Suppose also there exists a real number M such that $a_n > M$ for all positive integers n. Then $\{a_n\}$ converges, and $\lim_{n\to\infty} a_n \geq M$.

Example 7.23. Let n be a positive integer. Consider the sequence $a_n = \sqrt{n+1} - \sqrt{n}$. If $x > 0$, then the function $f(x) = \sqrt{x+1} - \sqrt{x}$ satisfies

$$f'(x) = \frac{1}{2}\left(\frac{1}{\sqrt{x+1}} - \frac{1}{\sqrt{x}}\right) = \frac{1}{2}\frac{\sqrt{x} - \sqrt{x+1}}{\sqrt{x(x+1)}} < 0.$$

Therefore, by the Fundamental Theorem of Calculus

$$a_{n+1} - a_n = f(n+1) - f(n) = \int_{x=n}^{x=n+1} f'(x)dx < 0.$$

That is, the sequence $\{a_n\}$ is decreasing. It is also positive, since $\sqrt{n+1} > \sqrt{n}$. We conclude that there exists a real number L such that $\lim_{n\to\infty} a_n = L$. In fact,

$$\lim_{n\to\infty}(\sqrt{n+1} - \sqrt{n}) = \lim_{n\to\infty} \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0.$$

7.3. **Infinite Series.** An **infinite series** is an infinite sum of the form $a_1 + a_2 + a_3 + \cdots$. Sometimes these series converge to some number, and sometimes they do not. For example

$$(1/2) + (1/4) + (1/8) + (1/16) + \cdots = 1.$$

We will now make this equality more precise.

Definition 7.24. Let $\{a_n\}$ be a sequence of numbers. We define

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$

And for any positive integer n, we define the Nth **partial sum** by

$$S_N = a_1 + \cdots + a_N = \sum_{n=1}^{N} a_n.$$

If the sequence of partial sums converges to a real number L, then we say the series **converges** and that its **sum** is L. In this case, we then write

$$a_1 + a_2 + \cdots = \sum_{n=1}^{\infty} a_n = L.$$

If the sequence of partial sums does not converges, we say that the series **diverges**. If the sequence of partial sums diverges to infinity, we say that the series **diverges to infinity**.

Example 7.25 (Geometric Series). Let r with $|r| < 1$. Let $a_n = r^n$. We will show that

$$S_N = \frac{r - r^{N+1}}{1-r}.$$

Indeed, we have

$$\begin{aligned} S_N &= r + r^2 + r^3 + \cdots + r^N. \\ rS_N &= \quad r^2 + r^3 + \cdots + r^N + r^{N+1}. \end{aligned}$$

Subtracting these two, we get

$$S_N - rS_N = r - r^{N+1}.$$

That is,

$$S_N = \frac{r - r^{N+1}}{1-r}.$$

Since $|r| < 1$, $\lim_{N\to\infty} r^{N+1} = 0$. So,

$$r + r^2 + r^3 + \cdots = \sum_{n=1}^{\infty} r^n = \frac{r}{1-r}.$$

In particular, using $r = 1/2$, we have

$$(1/2) + (1/4) + (1/8) + \cdots = \frac{1/2}{1/2} = 1.$$

Example 7.26. The following infinite sum was computed by Euler, though we will probably be unable to discuss this computation in this class.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Example 7.27. The sum $\sum_{n=1}^{\infty}(-1)^n$ diverges. To see this, note that $S_N = \sum_{n=1}^{N}(-1)^n = -1$ if N is odd, and $S_N = 0$ if N is even. That is, the sequence $\{S_N\}$ does not converge as $N \to \infty$. So, the sum $\sum_{n=1}^{\infty}(-1)^n$ diverges.

Example 7.28. The sum $\sum_{n=1}^{\infty} 1$ diverges. To see this, note that $S_N = \sum_{n=1}^{N} 1 = N$. That is, the sequence $\{S_N\}$ does not converge as $N \to \infty$. So, the sum $\sum_{n=1}^{\infty} 1$ diverges.

Example 7.29 (Telescoping Sum).

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

To sum the series, let $S_N = \sum_{n=1}^{N} \frac{1}{n(n+1)}$. Note that

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Therefore,

$$\begin{aligned} S_N &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1}\right) \\ &= \frac{1}{1} + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \cdots + \left(-\frac{1}{N} + \frac{1}{N}\right) - \frac{1}{N+1} \\ &= 1 - \frac{1}{N+1} \end{aligned}$$

So, $S_N \to 1$ as $N \to \infty$, as desired.

The following rules for infinite sums follow by applying the limits laws for sequences to the partial sum sequences $\{S_N\}$.

Proposition 7.30. *Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge. Let c be a real number. Then*

- $\sum_{n=1}^{\infty}(a_n + b_n) = (\sum_{n=1}^{\infty} a_n) + (\sum_{n=1}^{\infty} b_n)$.

- $\sum_{n=1}^{\infty}(a_n - b_n) = (\sum_{n=1}^{\infty} a_n) - (\sum_{n=1}^{\infty} b_n)$.
- $\sum_{n=1}^{\infty}(ca_n) = c(\sum_{n=1}^{\infty})$.

Example 7.31. Slightly generalizing our argument for the summation of the geometric series with $|r| < 1$, we have

$$\sum_{n=M}^{\infty} cr^n = \frac{cr^M}{1-r}.$$

If $|r| \geq 1$, then this series diverges.

Example 7.32.

$$\sum_{n=0}^{\infty} 7^{-n} = \frac{1}{1-(1/7)} = \frac{7}{6}.$$

$$\sum_{n=3}^{\infty} 2(-5)^{-n} = \frac{2(-1/5)^3}{1-(-1/5)} = \frac{-2/125}{6/5} = -\frac{1}{75}.$$

We now begin to develop different ways of testing whether or not a particular series converges or diverges. Such precise statements will be useful for us when we later investigate Taylor series.

Theorem 7.33 (**Divergence Test**). *Let $\{a_n\}$ be a sequence that does not converge to zero. Then $\sum_{n=1}^{\infty} a_n$ diverges.*

Remark 7.34. The contrapositive of this test says: If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$. Since the contrapositive is equivalent to the original statement, we can justify this test by demonstrating the contrapositive. If $\sum_{n=1}^{\infty} a_n$ converges, then the partial sums S_n converge as $n \to \infty$. Then, using the limit law, we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_n - S_{n-1}) = (\lim_{n\to\infty} S_n) - (\lim_{n\to\infty} S_{n-1}) = 0.$$

Example 7.35. The series $\sum_{n=1}^{\infty} \frac{4n+1}{3n}$ diverges, since the sequence $\{(4n+1)/(3n)\}$ converges as $n \to \infty$ to $(4/3) \neq 0$.

7.4. **Convergence of Series.**

7.4.1. *Convergence of Positive Series.*

In this section, we consider positive series. That is, we consider series $\sum a_n$ where $a_n > 0$ for all n. In this case, note that $S_{N+1} > S_N$, so the partial sums are strictly increasing. Recalling that increasing sequences either converge or diverge to infinity, we therefore have the following proposition.

Proposition 7.36 (**Dichotomy for Positive Series**). *Let $a_1, a_2, \ldots$ be a positive sequence, and let $S_N = \sum_{n=1}^{N} a_n$ be the N^{th} partial sum. Then exactly one of the two following cases holds.*

- *There exists $K > 0$ such that $S_N < K$ for all positive integers N. And in this case, $\sum_{n=1}^{\infty} a_n$ converges.*
- *For any $K > 0$, there exists $N = N(K)$ such that $S_N > K$. In this case, $\sum_{n=1}^{\infty} a_n$ diverges to infinity.*

Remark 7.37. This Dichotomy does not hold for all series. For example, $\sum_{n=1}^{\infty}(-1)^n$ diverges, but it also has bounded partial sums.

Here is one test that allows us to determine the convergence or divergence of positive series.

Theorem 7.38 (**Integral Test**). *Let $\{a_n\}$ be a positive sequence. Let f be a real valued function on the real line such that $a_n = f(n)$ for all positive integers n. Suppose f is positive, decreasing, and continuous on the domain $x \geq 1$.*

- *If $\int_1^\infty f(x)dx$ converges, then $\sum_{n=1}^\infty a_n$ converges.*
- *If $\int_1^\infty f(x)dx$ diverges, then $\sum_{n=1}^\infty a_n$ diverges.*

To justify the Integral Test, let $S_N = \sum_{n=1}^N a_n$ be the Nth partial sum. Since f is decreasing, we have $a_{n+1} = f(n+1) \leq f(x)$ for all $x \in [n, n+1]$. Therefore,

$$a_{n+1} \leq \int_n^{n+1} f(x)dx.$$

Summing up this inequality from $n = 2$ to $n = N$, we have

$$a_2 + a_3 + \cdots + a_N \leq \int_1^2 f(x)dx + \int_2^3 f(x)dx + \cdots + \int_{N-1}^N f(x)dx = \int_1^N f(x)dx.$$

Therefore,

$$S_N - a_1 \leq \int_1^N f(x)dx.$$

Since $\int_1^N f(x)dx$ increases to a finite number $K = \int_1^\infty f(x)dx$, we know that $S_N \leq a_1 + K$. So, from the dichotomy for positive series, $\sum_{n=1}^\infty a_n$ converges.

The other statement is proven similarly. Since f is decreasing, we have $a_n = f(n) \geq f(x)$ for all $x \in [n, n+1]$. Therefore,

$$a_n \geq \int_n^{n+1} f(x)dx.$$

Summing up this inequality from $n = 1$ to $n = N$, we have

$$a_1 + a_2 + \cdots + a_N \geq \int_1^2 f(x)dx + \int_2^3 f(x)dx + \cdots + \int_N^{N+1} f(x)dx = \int_1^{N+1} f(x)dx.$$

Therefore,

$$S_N \geq \int_1^{N+1} f(x)dx.$$

Since $\int_1^{N+1} f(x)dx$ diverges to infinity, we know that S_N also diverges to infinity.

Example 7.39 (**Harmonic Series**). The harmonic series $\sum_{n=1}^\infty 1/n$ diverges to infinity. This follows since $\int_1^\infty (1/x)dx$ also diverges to infinity.

Example 7.40. Let p be a real number. Then $\sum_{n=1}^\infty \frac{1}{n^p}$ converges if and only if $p > 1$.

If $p > 1$, then the convergence follows from our known convergence of the integral $\int_1^\infty x^{-p}dx$. The case $p = 1$ was just dealt with. If $p \leq 0$, then the terms n^{-p} do not converge to zero, so their sum diverges by the Divergence Test. The only remaining case is $0 < p < 1$. In this case, the sum diverges since the integral $\int_1^\infty x^{-p}dx$ also diverges.

Theorem 7.41 (**Comparison Test**). *Assume there exists a positive integer M such that $0 \leq a_n \leq b_n$ for all $n \geq M$.*

- *If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.*
- *If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.*

This Theorem follows pretty quickly from the Dichotomy for positive series. Since the convergence or divergence of a series is the same when we change a finite number of terms of the series, we may assume that $M = 1$. In this case, the Nth partial sum of $\{b_n\}$ is larger than the Nth partial sum of $\{a_n\}$. If $\sum_{n=1}^{\infty} b_n$ converges, then there exists $K > 0$ such that all partial sums of b_n are bounded by K. So, all partial sums of a_n are bounded by K, and therefore $\sum_{n=1}^{\infty} a_n$ converges, by the Dichotomy for positive series. If $\sum_{n=1}^{\infty} a_n$ diverges, then its partial sums diverge to infinity (since the series is positive). So, the partial sums of b_n diverge to infinity. So, the $\sum_{n=1}^{\infty} b_n$ diverges.

Example 7.42. We demonstrate that $\sum_{n=1}^{\infty} \frac{1}{2^n\sqrt{n}}$ converges. Note that $0 \leq 1/(2^n\sqrt{n}) \leq 1/2^n$ for all positive integers n. And $\sum_{n=1}^{\infty} 1/2^n$ converges. So, $\sum_{n=1}^{\infty} \frac{1}{2^n\sqrt{n}}$ converges.

Example 7.43. We show that $\sum_{n=1}^{\infty} \frac{1}{(n^2+3)^{1/2}}$ diverges. Note that $0 \leq (n^2+3)^{1/2} \leq (2n^2)^{1/2} \leq 2n$ for every integer $n \geq 2$. Therefore, $(n^2+3)^{-1/2} \geq 1/(2n) > 0$ for every integer $n \geq 2$. Since $\sum_{n=1}^{\infty} 1/(2n)$ diverges, we conclude that $\sum_{n=1}^{\infty} \frac{1}{(n^2+3)^{1/2}}$ diverges.

The following Convergence test follows readily from the Comparison Test.

Theorem 7.44 (Limit Comparison Test). *Let $\{a_n\},\{b_n\}$ be positive sequences. Assume that the following limit exists:*

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}.$$

- *If $0 < L < \infty$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.*
- *If $L = \infty$, and if $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges.*
- *If $L = 0$, and if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.*

Proof. Assume first that $L > 0$. Then there exists M, R and ε such that $0 < \varepsilon \leq a_n/b_n < R$ for all $n > M$. If $\sum_{n=1}^{\infty} a_n$ converges, then since $0 \leq b_n \leq a_n/\varepsilon$, the Comparison Test shows that $\sum_{n=1}^{\infty} b_n$ converges. If $\sum_{n=1}^{\infty} b_n$ converges, then since $0 \leq a_n \leq b_n R$, the Comparison Test shows that $\sum_{n=1}^{\infty} a_n$ converges.

Assume now that $L = \infty$. Then there exists M and ε such that $0 < \varepsilon \leq\leq a_n/b_n$ for all $n > M$. If $\sum_{n=1}^{\infty} a_n$ converges, then since $0 \leq b_n \leq a_n/\varepsilon$, the Comparison Test shows that $\sum_{n=1}^{\infty} b_n$ converges.

Assume now that $L = 0$. Then there exists M and R such that $0 \leq a_n/b_n < R$ for all $n > M$. If $\sum_{n=1}^{\infty} b_n$ converges, then since $0 \leq a_n \leq b_n R$, the Comparison Test shows that $\sum_{n=1}^{\infty} a_n$ converges. □

Example 7.45. The series $\sum_{n=1}^{\infty} \frac{n^2}{n^4+3n+5}$ converges. To see this, let $a_n = n^2/(n^4+3n+5)$, and let $b_n = 1/n^2$. We know that $\sum_{n=1}^{\infty} b_n$ converges to $\pi^2/6$. Also, note that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^4}{n^4+3n+5} = 1.$$

Therefore, $\sum_{n=1}^{\infty} a_n$ converges, by the Limit Comparison Test.

7.4.2. *Conditional Convergence.*

Definition 7.46 (**Absolute Convergence**)**.** We say that the series $\sum_{n=1}^{\infty} a_n$ **converges absolutely** if and only if the series $\sum_{n=1}^{\infty} |a_n|$ converges.

Example 7.47. The series $\sum_{n=1}^{\infty}(-1)^n/n^2$ converges absolutely, since $\sum_{n=1}^{\infty} 1/n^2$ converges.

Proposition 7.48. *If a series converges absolutely, then it converges. That is: if $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.*

To prove this proposition, note that $-|a_n| \leq a_n \leq |a_n|$. So, adding $|a_n|$ to both sides,

$$0 \leq a_n + |a_n| \leq 2|a_n|.$$

We know that $\sum 2|a_n|$ converges, so by the Comparison Test, $\sum(a_n + |a_n|)$ converges. Since $\sum |a_n|$ converges, we can subtract these two convergent series to get another convergent series. That is, $\sum[(a_n + |a_n|) - |a_n|] = \sum a_n$ converges.

Example 7.49. The series $\sum_{n=1}^{\infty}(-1)^n/n^2$ converges absolutely, since $\sum_{n=1}^{\infty} 1/n^2$ converges. Therefore, we also know that the series $\sum_{n=1}^{\infty}(-1)^n/n^2$ converges.

Definition 7.50 (**Conditional Convergence**)**.** We say that a series $\sum a_n$ **converges conditionally** if and only if $\sum a_n$ converges, but $\sum |a_n|$ diverges.

Remark 7.51. If a series converges conditionally, then it does not converge absolutely.

Example 7.52. The series $\sum_{n=1}^{\infty}(-1)^{n-1}/n$ does not absolutely converge, since the harmonic series $\sum_{n=1}^{\infty} 1/n$ diverges. However, $\sum_{n=1}^{\infty}(-1)^{n-1}/n$ does converge. So, $\sum_{n=1}^{\infty}(-1)^{n-1}/n$ is conditionally convergent. To see this, consider the Nth partial sum. When n is even, $(-1)^{n-1}/n$ is negative, and when n is odd, $(-1)^{n-1}/n$ is positive. So,

$$S_{2N} + 1/(2N+1) - 1/(2N+2) = S_{2N+2}.$$

And since $1/(2N+1) - 1/(2N+2) > 0$, we see that

$$S_{2N+2} > S_{2N}.$$

Written another way,

$$S_2 < S_4 < S_6 < \cdots \qquad (*)$$

Arguing similarly,

$$S_{2N+1} - 1/(2N+2) + 1/(2N+3) = S_{2N+3}.$$

And since $-1/(2N+2) + 1/(2N+3) < 0$, we have

$$S_{2N+3} < S_{2N+1}.$$

That is,

$$S_1 > S_3 > S_5 > \cdots \qquad (**)$$

Now, $S_{2N} + 1/(2N+1) = S_{2N+1}$, so

$$S_{2N} < S_{2N+1} \qquad (\ddagger)$$

We claim that any even partial sum is bounded by every odd partial sum. To see this, note that, for any positive integer K,

$$S_{2N} \overset{(*)}{<} S_{2N+2} < \cdots < S_{2(N+K)} \overset{(\ddagger)}{<} S_{2(N+K)+1} \overset{(**)}{<} S_{2(N+K)-1} < S_{2(N+K)-3} < \cdots < S_3 < S_1.$$

Combining this observation with ($*$) and ($**$), we therefore have the following inequalities.

$$0 < S_2 < S_4 < S_6 < \cdots < S_7 < S_5 < S_3 < S_1.$$

So, the even partial sums are increasing and bounded from above, and the odd partial sums are decreasing and bounded from below. Both of these partial sums therefore converge to (possibly different) limits L and L'. However,

$$L - L' = \lim_{N\to\infty} S_{2N} - \lim_{N\to\infty} S_{2N+1} = \lim_{N\to\infty}(S_{2N} - S_{2N+1}) = \lim_{N\to\infty}(-1/(2N+1)) = 0.$$

Therefore, $L = L'$. In conclusion, the series $\sum_{n=1}^{\infty}(-1)^{n-1}/n$ converges.

Actually, the argument we just provided generalizes to give the following.

Theorem 7.53 (**Leibniz Test for Alternating Series**). *Suppose $\{a_n\}$ is a positive decreasing sequence that converges to zero. That is,*

$$a_1 > a_2 > a_3 > \cdots > 0, \qquad \lim_{n\to\infty} a_n = 0.$$

Then the following alternating series converges.

$$S = \sum_{n=1}^{\infty}(-1)^{n-1}a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$

Moreover, for all integers $N \geq 1$,

$$0 < S < a_1, \qquad S_{2N} < S < S_{2N+1}, \qquad |S - S_N| < a_{N+1}.$$

Example 7.54. The series $\sum_{n=1}^{\infty}(-1)^{n-1}/\sqrt{n}$ converges to a number S with $0 \leq S \leq 1$. However, recall that $\sum_{n=1}^{\infty} 1/\sqrt{n}$ diverges by the integral test. So, $\sum_{n=1}^{\infty}(-1)^{n-1}/\sqrt{n}$ converges conditionally.

Remark 7.55. Conditionally convergent series are more subtle than absolutely convergent series in the following way. If we rearrange the terms of an absolutely convergent series, then the resulting series still converges and has the same sum. However, if we rearrange the terms of a conditionally convergent series, then we can make the series converge to *any number* or we can make the series diverge. To see this, consider the conditionally convergent series $\sum_{n=1}^{\infty}(-1)^{n-1}/n$. When n is odd, the terms are positive, and when n is even, the terms are negative. Also, it follows from the integral test that summing only the odd terms will give a divergent series. So, let's first sum the odd terms until we get a sum larger than 100.

$$1/1 + 1/3 + 1/5 + 1/7 + \cdots + 1/N > 100.$$

Now, let's sum the first negative term. All terms are always less than 1 in absolute value, so

$$1/1 + 1/3 + 1/5 + \cdots + 1/N - 1/2 > 99.$$

Now, let's sum positive terms again until we get larger than 200.

$$1/1 + 1/3 + 1/5 + \cdots + 1/N - 1/2 + 1/(N+2) + 1/(N+4) + \cdots + 1/(M) > 200.$$

Now, let's sum the second negative term.

$$1/1 + 1/3 + 1/5 + \cdots + 1/N - 1/2 + 1/(N+2) + 1/(N+4) + \cdots + 1/(M) - 1/4 > 199.$$

And now, let's sum positive terms again until we go larger than 300. And so on. Continuing in this way, every term will appear, and the partial sums will grow arbitrarily large.

7.5. **Root and Ratio Tests.** We now investigate a few more tests that will be quite useful in our further investigations below.

Theorem 7.56 (**Ratio Test**)**.** *Let $\{a_n\}$ be a sequence. Assume that the following limit exists:*

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

- *If $\rho < 1$ then $\sum_{n=1}^{\infty} a_n$ converges absolutely.*
- *If $\rho > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.*
- *If $\rho = 1$, then the test is inconclusive. That is, the sum may converge or diverge.*

Suppose $\rho < 1$. Then, there exists M and $0 < r < 1$ such that, for all $n \geq M$, $|a_{n+1}| < r\,|a_n|$. So,

$$|a_{j+M}| < r\,|a_{j-1+M}| < r^2\,|a_{j-2+M}| < \cdots < r^j\,|a_M|\,.$$

Therefore,

$$\left|\sum_{n=M}^{\infty} a_n\right| \leq \sum_{n=M}^{\infty} |a_n| \leq \sum_{n=M}^{\infty} r^{n-M}\,|a_M| = |a_M|\,\frac{1}{1-r} < \infty.$$

So, if $\rho < 1$, $\sum a_n$ converges.

Suppose $\rho > 1$. The argument is similar to before. There exists M and $r > 1$ such that, for all $n \geq M$, $|a_{n+1}| > r\,|a_n| > 0$. So,

$$|a_{j+M}| > r\,|a_{j-1+M}| > r^2\,|a_{j-2+M}| > \cdots > r^j\,|a_M| > 0.$$

So, a_n does not converge to zero as $n \to \infty$. By the Divergence Test, $\sum a_n$ must diverge.

Finally, we present an example that shows the case $\rho = 1$ is inconclusive.

Example 7.57. The sum $\sum_{n=1}^{\infty} n$ diverges, but $\rho = 1$ in the Ratio Test, since $\lim_{n\to\infty} \left|\frac{n+1}{n}\right| = 1$.

The sum $\sum_{n=1}^{\infty} n^{-2}$ converges, but $\rho = 1$ in the Ratio Test, since $\lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \frac{n^2}{n^2+2n+1} = 1$.

Example 7.58. Let x be any real number. We will show that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges. Therefore, there is a sensible meaning to the expression

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Note that this series converges for $x = 0$, so let $x \neq 0$. Using the Ratio Test, we get

$$\rho = \lim_{n\to\infty} \left|\frac{x^{n+1} n!}{(n+1)! x^n}\right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0.$$

The following test is closely related to the Ratio Test. (In fact, the proof of the Root Test is similar to the proof of the Ratio Test.)

Theorem 7.59 (**Root Test**)**.** *Let $\{a_n\}$ be a sequence. Assume that the following limit exists:*

$$L = \lim_{n\to\infty} |a_n|^{1/n}$$

- *If $L < 1$ then $\sum a_n$ converges absolutely.*
- *If $L > 1$, then $\sum a_n$ diverges.*

- *If $L = 1$, then the test is inconclusive. That is, the sum may converge or diverge.*

Remark 7.60. The Ratio Test is often easier to use than the Root Test, though the Root Test is technically the stronger statement.

Example 7.61. The sum $\sum_{n=1}^{\infty} \left(\frac{n}{2n+4}\right)^n$ converges, since $L = \lim_{n\to\infty} \frac{|n|}{|2n+4|} = (1/2) < 1$.

7.6. **Power Series.** We are finally ready to begin the discussion of infinite sums of functions, which will be the culmination of the course. As we have discussed above, some of the most useful functions have expression in terms of infinite sums of monomials. Also, some presumably complicated functions can be better understood by their expression as an infinite sum of monomials. For example, in the previous section we showed that the following power series converges for all x.

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

We will show in the next section that this power series is actually equal to e^x. So, even though near the beginning of the course, the definition of the exponential function may have been a bit cumbersome, we now have an arguably simpler way of describing this function. Using the Ratio Test, we can similarly conclude that, for all x the following series converges

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}(-1)^n}{(2n+1)!}$$

This function of x turns out to be equal to $\sin(x)$. Using the Ratio Test, we can similarly conclude that, for all x the following series converges

$$\sum_{n=0}^{\infty} \frac{x^{2n}(-1)^n}{(2n)!}$$

This function of x turns out to be equal to $\cos(x)$.

However, as we have seen from our convergence tests, not all infinite series converge. A great example of this phenomenon is the infinite series for $|x| < 1$.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

- When $|x| < 1$, the series converges, and both sides are equal.
- When $x = 1$, the left side is undefined and the right side diverges.
- When $x = -1$, the left side is defined, but the right side still diverges.
- When $|x| > 1$, the left side is defined, but the right side diverges.

So, even though we have a nice expression for the infinite series $\sum_{n=0}^{\infty} x^n$ when $|x| < 1$, there may be no meaning to this infinite series when x is too large. In this case, we say that the radius of convergence of the power series $\sum_{n=0}^{\infty} x^n$ is 1. We now make these statements more precise.

Definition 7.62 (**Power Series**). Let x be a real variable, and let $\{a_n\}$ be a sequence. A **power series** with center c is an infinite series of the form

$$F(x) = \sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

Any power series behaves analogously to the power series for $1/(1-x)$ as follows.

Theorem 7.63 (Radius of Convergence). *Suppose we have a power series with center c:*

$$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n.$$

Then there is a number R such that $0 \leq R \leq \infty$ which is a radius of convergence for F. That is, if R is finite, then $F(x)$ converges absolutely whenever $|x-c| < R$ and $F(x)$ diverges whenever $|x-c| > R$. (When $|x-c| = R$, convergence or divergence can occur.) If R is infinite, then the power series converges absolutely for all real x.

To see this result for $c = 0$, note that if $F(y)$ converges, then $F(x)$ converges absolutely for all x with $|x| < |y|$. To see this, note that since $\sum_{n=0}^{\infty} a_n y^n$ converges, the quantity $a_n y^n$ converges to zero (by the Divergence Test). In particular, there exists an $M > 0$ such that $|a_n y^n| < M$ for all n. Then

$$|F(x)| \leq \sum_{n=0}^{\infty} |a_n| \, |x|^n \leq \sum_{n=0}^{\infty} |a_n| \, |y|^n \, \frac{|x|^n}{|y|^n} \leq \sum_{n=0}^{\infty} M \frac{|x|^n}{|y|^n}.$$

Since $|x| < |y|$, we have $|x| \,/\, |y| < 1$, so the final series is convergent, being a geometric series. In conclusion $F(x)$ is absolutely convergent. So, if F converges for any y, then F converges absolutely for all x with $|x| < |y|$. So, if we let R be the largest number such that F converges on $(-R, R)$, then the Theorem is proven.

Example 7.64. We saw above that the power series $\sum_{n=0}^{\infty} x^n$ (with $c = 0$) has radius of convergence $R = 1$. In this case, the series diverges when $|x| = 1$, and the series diverges when $|x| > 1$.

Example 7.65. Recall that $1/(1-x) = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$. We can find the power series of other functions by substituting monomials into this function. For example,

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty} 2^n x^n$$

This power series has radius of convergence $R = 1/2$.

We now describe another way to find power series. Given a power series with a positive radius of convergence, it turns out that we can differentiate the series term by term, as follows.

Proposition 7.66 (Term-by-Term Differentiation and Integration). *Suppose we have a power series F with center c and with radius of convergence $R > 0$.*

$$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n.$$

If R is finite, then F is differentiable on $(c-R, c+R)$. If R is infinite, then F is differentiable everywhere. Moreover, for any $x \in (c-R, c+R)$, we can differentiate and integrate F term-by-term to get new power series also with radius of convergence R:

$$F'(x) = \sum_{n=0}^{\infty} n a_n (x-c)^{n-1}.$$

$$\int F(x)dx = A + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1}.$$

Example 7.67. The power series $1/(1-x) = \sum_{n=0}^{\infty} x^n$ has radius of convergence $R = 1$. By differentiating both sides, we get a new power series with radius of convergence $R = 1$:

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots .$$

Example 7.68. The inverse tangent function satisfies $(\tan^{-1}(x))' = 1/(1+x^2)$. Plugging in $-x^2$ into the series expansion of $1/(1-x)$, we have

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-1)^n x^{2n}.$$

We can check that this series has radius of convergence $R = 1$. So, integrating this series term-by-term gives a power series for inverse tangent with a radius of convergence $R = 1$.

$$\tan^{-1}(x) = A + \int \frac{1}{1+x^2}dx = A + \sum_{n=0}^{\infty} \int (-1)^n x^{2n} dx = A + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}.$$

Since $\tan^{-1}(0) = 0$, we have $A = 0$. So, we have the following power series with radius of convergence $R = 1$.

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}.$$

In this case, the power series converges at the point $x = 1$, by the alternating series test. We therefore get an infinite sum formula for π:

$$\frac{\pi}{4} = \tan^{-1}(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

We will soon conclude the course by discussing a general method for finding power series expansions of various functions. For now, let's just see a special method that allows us to find the infinite series expansion of the exponential function. Consider the power series expansion

$$F(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

From the ratio test, we saw that this series converges for all x. That is, the radius of convergence R satisfies $R = \infty$. So, we can differentiate this series term by term, to get another power series whose radius of convergence is infinite.

$$F'(x) = \sum_{n=0}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = F(x).$$

That is, $F'(x) = F(x)$. Note also that $F(0) = 1$. Recall that this differential equation is only satisfied by e^x, therefore $F(x) = e^x$. The argument is the following:

$$\frac{d}{dx}(e^{-x}F(x)) = e^{-x}F'(x) - e^{-x}F(x) = e^{-x}(F'(x) - F(x)) = 0.$$

Therefore, $e^{-x}F(x) = C$ for some constant C. Since $F(0) = e^0 = 1$, we have $C = 1$, so that $F(x) = e^x$, as desired.

7.7. **Taylor Series.** Now, in the culmination of the course, we will identify the power series expansions of various special functions. We have already shown, using a differentiation argument, that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad -\infty < x < \infty.$$

We now present a more general method for finding power series expansions. This method is closely related to our discussion of Taylor Series.

Theorem 7.69 (**Uniqueness of Taylor Series**). *Suppose that for all x with $|x - c| < R$, we can write a function f as a convergent power series*

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n.$$

Then the coefficients a_n are given by

$$a_n = \frac{f^{(n)}(c)}{n!}.$$

That is, f is equal to its ***Taylor series*** *for $|x - c| < R$. In the case $c = 0$, we say that f is equal to its* ***Maclaurin series****.*

To prove this Theorem, first note that $f(c) = a_0$. Then, from the term-by-term differentiation theorem, we can always differentiate the power series $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ when $|x - c| < R$. In particular, taking one derivative, we have

$$f'(c) = \sum_{n=0}^{\infty} n a_n (c-c)^{n-1} = a_1.$$

Taking more derivatives, we have

$$f''(c) = \sum_{n=0}^{\infty} n(n-1) a_n (c-c)^{n-2} = 2! a_2.$$

$$f'''(c) = \sum_{n=0}^{\infty} n(n-1)(n-2) a_n (c-c)^{n-3} = 3! a_3.$$

And so on.

Note that in the above Theorem, it is given that f is equal to some power series expansion. However, we would really like to start with a function, and then determine that the function is equal to its Taylor series. Therefore, we need to find some condition on a function that guarantees that the function is equal to its Taylor series. Note that in general, even if a function is infinitely differentiable, it may **not** be equal to its Taylor series. Here is an example.

Example 7.70. Consider the function

$$f(x) = \begin{cases} e^{-1/x^2} & \text{, if } x \neq 0 \\ 0 & \text{, if } x = 0 \end{cases}.$$

This function can be shown to be infinite differentiable. However, it follows from L'Hôpital's rule that $f^{(n)}(0) = 0$ for all positive integers n. Therefore, the Taylor expansion of f is just the zero function. However, f is a nonzero function. In fact, f is only zero when $x = 0$. So, f is not equal to its Taylor series, except at the point $x = 0$.

Theorem 7.71 (Main Theorem of the Course). *Let $R > 0$. Let f be an infinitely differentiable function. Let $K > 0$ so that*

$$|f^n(x)| \leq K \textit{ for all } n \geq 0 \textit{ and for all } c - R < x < c + R.$$

Then for all $c - R < x < c + R$, we know that f is equal to its Taylor series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n \textit{ for all } c - R < x < c + R.$$

Let T_n be the nth degree Taylor polynomial of f at $x = c$. To prove this Theorem, recall that the Error Bound for Taylor Polynomials says that

$$|f(x) - T_n(x)| \leq K \frac{|x-c|^{n+1}}{(n+1)!}$$

So, if $|x - c| < R$, then

$$|f(x) - T_n(x)| \leq K \frac{R^{n+1}}{(n+1)!} \to 0 \quad \text{as } n \to \infty.$$

(Recall that the Ratio Test implies $\lim_{n\to\infty} R^{n+1}/(n+1)! = 0$.) In conclusion, $f(x) = \lim_{n\to\infty} T_n(x)$, as desired.

Example 7.72. The sine and cosine functions are both infinitely differentiable, with all derivatives bounded in absolute value by 1 for all $-\infty < x < \infty$. So, the above Theorem implies that they are equal to their Maclaurin series, for all real x.

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

Similarly, for any fixed radius R, the exponential function e^x has all derivatives bounded by e^R for $-R < x < R$. So, the exponential function is equal to its Maclaurin series for any fixed R. So, the exponential function is equal to its Maclaurin series for all real x. (Though we knew this already by a differentiation argument.)

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Here are some other ways to find Taylor series of various functions, using what we already know.

Example 7.73.

$$x^2 e^x = x^2 \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!}.$$

Example 7.74 (**Multiplication**).

$$\begin{aligned}
e^{2x} &= \sum_{n=0}^{\infty}\frac{(2x)^n}{n!} = e^x e^x = (\sum_{n=0}^{\infty}\frac{x^n}{n!})(\sum_{m=0}^{\infty}\frac{x^m}{m!})\\
&= (1+x+x^2/2!+\cdots)(1+x+x^2/2!+\cdots)\\
&= 1+x(1+1)+x^2(1+2/2!)+x^3(2/2!+2/3!)+x^4(2/4!+2/3!+1/(2!)^2)+\cdots\\
&= 1+2x+2x^2+(4/3)x^3+(2/3)x^4+\cdots
\end{aligned}$$

Note that we have rearranged an infinite sum. So, we are using the fact that these series converge absolutely. Recall that if a series converges only conditionally, then we **cannot** rearrange the series and still get the same sum.

In general, if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$, and both series converge for $|x| < R$, then they both converge absolutely for $|x| < R$. And the following series converges absolutely for $|x| < R$.

$$f(x)g(x) = \sum_{n=0}^{\infty} c_n x^n.$$

where

$$c_n = \sum_{j=0}^{n} a_j b_{n-j} = a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0.$$

Example 7.75 (**Integration**). We can use infinite series to get Taylor series expressions for integrals that cannot be evaluated explicitly. For example,

$$\begin{aligned}
\int_0^1 \sin(x^2)dx &= \int_0^1 \sum_{n=0}^{\infty}\frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\int_0^1 x^{4n+2}dx\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\frac{1}{4n+3} = \frac{1}{3}-\frac{1}{42}+\frac{1}{1320}-\frac{1}{75600}+\cdots.
\end{aligned}$$

Example 7.76. Let a be any real number (which is not necessarily an integer). We will prove Newton's generalization of the binomial series formula. Define

$$\binom{a}{n} = \frac{a(a-1)(a-2)\cdots(a-n+1)}{n!}. \qquad \binom{a}{0} = 1.$$

Then for any $|x| < 1$,

$$(1+x)^a = \sum_{n=0}^{\infty}\binom{a}{n}x^n = F(x).$$

In the case that a is a positive integer, this sum is finite. However, in all other cases, this sum is infinite. The Ratio Test shows that (if a is not a positive integer),

$$\binom{a}{n+1}\Big/\binom{a}{n} = \frac{a(a-1)\cdots(a-n)}{(n+1)!}\frac{n!}{a(a-1)\cdots(a-n+1)} = \frac{a-n}{n}.$$

So, the radius of convergence of the Taylor series is $R = 1$. To see that the function $(1+x)^a$ is equal to its Taylor series $F(x)$ for $|x| < 1$, one can show that

$$\frac{d}{dx}\frac{\sum_{n=0}^{\infty}\binom{a}{n}x^n}{(1+x)^a} = 0.$$

Indeed,

$$F'(x) = \sum_{n=0}^{\infty} n\binom{a}{n}x^{n-1} = \sum_{n=1}^{\infty} n\binom{a}{n}x^{n-1} = \sum_{n=1}^{\infty}(a-n+1)\binom{a}{n-1}x^{n-1} = \sum_{n=0}^{\infty}(a-n)\binom{a}{n}x^n$$

$$\begin{aligned}(1+x)F'(x) = F'(x) + xF'(x) &= \sum_{n=0}^{\infty}(a-n)\binom{a}{n}x^n + \sum_{n=0}^{\infty} n\binom{a}{n}x^n \\ &= \sum_{n=0}^{\infty}\left((a-n)+n\right)\binom{a}{n}x^n \\ &= \sum_{n=0}^{\infty} a\binom{a}{n}x^n = a\left(\sum_{n=0}^{\infty}\binom{a}{n}x^n\right) = aF(x).\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}\frac{\sum_{n=0}^{\infty}\binom{a}{n}x^n}{(1+x)^a} &= \frac{d}{dx}\frac{F(x)}{(1+x)^a} = \frac{(1+x)^aF'(x) - aF(x)(1+x)^{a-1}}{(1+x)^{2a}} \\ &= \frac{(1+x)^{a-1}[(1+x)F'(x) - aF(x)]}{(1+x)^{2a}} = 0.\end{aligned}$$

Combining this fact with $F(0) = 1$ and $(1+0)^a = 1$, we get the identity

$$(1+x)^a = \sum_{n=0}^{\infty}\binom{a}{n}x^n, \qquad |x| < 1.$$

Exercise 7.77. The following integral often arises in probability theory, in relation to diffusions, Brownian motion, and so on.

$$F(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt.$$

Using a Taylor series for e^{-t^2}, find a Taylor series for F. Then, find the radius of convergence of this series. Finally, compute $F(1/\sqrt{2})$ to four decimal places of accuracy. (F is also known as a bell curve, or the error function.)

Exercise 7.78. Let $i = \sqrt{-1}$. Using the Maclaurin series for $\sin(x)$, $\cos(x)$ and e^x, verify Euler's identity

$$e^{ix} = \cos(x) + i\sin(x).$$

In particular, using $x = \pi$, we have

$$e^{i\pi} + 1 = 0.$$

Also, use Euler's identity to prove the following equalities

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}.$$

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}.$$

In particular, we finally see that the hyperbolic sine and cosine functions are exactly the usual sine and cosine functions, evaluated on imaginary numbers.

$$\cos(x) = \cosh(ix).$$
$$\sin(x) = \sinh(ix)/i.$$

Exercise 7.79. Euler's identity can be used to remember all of the multiple angle formulas that are easy to forget. For example, note that

$$\cos(2x) + i\sin(2x) = e^{2ix} = (e^{ix})^2 = (\cos(x) + i\sin(x))^2 = \cos^2(x) - \sin^2(x) + 2i\sin(x)\cos(x).$$

By equating the real and imaginary parts of this identity, we therefore get

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$
$$\sin(2x) = 2\sin(x)\cos(x).$$

Derive the triple angle identities in this same way, using $e^{3ix} = (e^{ix})^3$.

8. Appendix: Notation

Let x be a real number. Let $a > 0$

$\mathbb{R}$ denotes the set of real numbers

$\in$ means "is an element of." For example, $2 \in \mathbb{R}$ is read as "2 is an element of $\mathbb{R}$."

$f\colon A \to B$ means f is a function with domain A and range B. For example,

$f\colon [0,1] \to \mathbb{R}$ means that f is a function with domain $[0,1]$ and range $\mathbb{R}$

e^x denotes the exponential function

$\ln(x)$ denotes the natural logarithm of $x > 0$, i.e. the inverse of the exponenial function

$\log_a(x) = \ln(x)/\ln(a)$

$\sin^{-1}(x)$ denotes the inverse of $\sin\colon [-\pi/2, \pi/2] \to [-1,1]$

$\cos^{-1}(x)$ denotes the inverse of $\cos\colon [0,\pi] \to [-1,1]$

$\tan^{-1}(x)$ denotes the inverse of $\tan\colon [-\pi/2, \pi/2] \to (-\infty, \infty)$

$\cosh(x) = (e^x + e^{-x})/2$

$\sinh(x) = (e^x - e^{-x})/2$

UCLA Department of Mathematics, Los Angeles, CA 90095-1555
E-mail address: heilman@math.ucla.edu