Please provide complete and well-written solutions to the following exercises.

Due December 4, at the beginning of class.

Assignment 8

Exercise 1. Find the average value of the function $h(x) = (\cos(x))^4 \sin(x)$ on the interval $[0, \pi]$.

Exercise 2. Let $a > 0$. Evaluate $\int_0^a x\sqrt{a^2 - x^2}dx$.

Exercise 3. State whether or not the statement is True or False. Justify your answer. Let $a < b$.

(1) If $f, g\colon [a, b] \to \mathbf{R}$ are continuous, then
$$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$
(2) If $f, g\colon [a, b] \to \mathbf{R}$ are continuous, then
$$\int_a^b (f(x)g(x))dx = \left(\int_a^b f(x)dx\right)\left(\int_a^b g(x)dx\right).$$
(3) If $f, g\colon [a, b] \to \mathbf{R}$ are continuous, and if $f(x) \geq g(x)$ for all $x \in [a, b]$, then
$$\int_a^b f(x)dx \geq \int_a^b g(x)dx.$$
(4) If $f, g\colon [a, b] \to \mathbf{R}$ are continuous, and if $f(x) > g(x)$ for all $x \in [a, b]$, then
$$\int_a^b f(x)dx > \int_a^b g(x)dx.$$
(5) If $f\colon [a, b] \to \mathbf{R}$ is continuous, then f has an antiderivative on $[a, b]$.
(6) If $f\colon (a, b) \to \mathbf{R}$ is continuous, then $\int_a^b f(x)dx$ exists.

Exercise 4. What is $\lim_{h\to 0} \frac{1}{h}\int_2^{2+h} \sin(x^2)dx$

Exercise 5. What is $\lim_{h\to 0} \int_2^{2+h} \sin(x^2)dx$?

Exercise 6. Compute the area between the curves $x = y^2 - 4y$ and $x = 2y - y^2$.

Exercise 7. Suppose $f\colon \mathbf{R} \to \mathbf{R}$ is continuous and
$$x\sin(\pi x) = \int_0^{x^2} f(t)dt$$
Find $f(4)$.

Exercise 8. A high-tech company purchases a new computing system whose initial value is V. The system will depreciate at the rate $f = f(t)$ and will accumulate maintenance costs at the rate $g = g(t)$, where t is the time measure in months. The company wants to determine the optimal time to replace the system.

(a) Let
$$C(t) = \frac{1}{t}\int_0^t [f(s) + g(s)]ds.$$
Show that the critical points of C occur at the numbers t where $C(t) = f(t) + g(t)$.

(b) Suppose
$$f(t) = \begin{cases} \frac{V}{15} - \frac{V}{450}t & \text{, if } 0 < t \leq 30 \\ 0 & \text{, if } t > 30 \end{cases},$$
and suppose $g(t) = \frac{Vt^2}{12900}$ for $t > 0$. Determine the length of time T for the total depreciation $D(t) = \int_0^t f(s)ds$ to equal the initial value V.

(c) Determine the absolute minimum of C on $(0, T]$.

(d) Sketch the graphs of C and $f+g$ in the same coordinate system, and verify the result of part (a) in this case.

Exercise 9. (Optional challenge question, ungraded) The following formula comes from Chapter 6, but it is so useful that it should be mentioned. This formula allows us to move around a derivative inside an integral. Let $f, g\colon [a, b] \to \mathbf{R}$ be differentiable functions. Use the Fundamental Theorem of Calculus and the product rule to derive the **integration by parts formula**:
$$\int_a^b f'(x)g(x)dx = [f(x)g(x)]_{x=a}^{x=b} - \int_a^b f(x)g'(x)dx$$
$$= [f(b)g(b) - f(a)g(a)] - \int_a^b f(x)g'(x)dx.$$