Please provide complete and well-written solutions to the following exercises.

Due March 12, in the discussion section.

Assignment 8

Exercise 1. Let E be a subset of $\mathbf{R}$, let $f\colon E\to\mathbf{R}$, let $x_0\in E$, and let $L\in\mathbf{R}$. Show that the following two statements are equivalent.

- f is differentiable at x_0 and $f'(x_0)=L$.
- We have $\lim_{x\to x_0; x\in E\setminus\{x_0\}}\frac{|f(x)-(f(x_0)+L(x-x_0))|}{|x-x_0|}=0$.

Exercise 2. Let E be a subset of $\mathbf{R}^n$, let $f\colon E\to\mathbf{R}^m$ be a function, and let x_0 be an *interior point* of E. Let $L_a\colon\mathbf{R}^n\to\mathbf{R}^m$ and let $L_b\colon\mathbf{R}^n\to\mathbf{R}^m$ be linear transformations. Suppose f is differentiable at x_0 with derivative L_a, and f is differentiable at x_0 with derivative L_b. Show that $L_a=L_b$. (Hint: argue by contradiction. Assume that $L_a\neq L_b$. Then there exists a nonzero vector $v\in\mathbf{R}^n$ such that $L_av\neq L_bv$. Then, apply the definition of the derivative, and try to specialize to the case where $x=x_0+tv$ for some scalar t, in order to obtain a contradiction.)

Exercise 3. Let E be a subset of $\mathbf{R}^n$, let $f\colon E\to\mathbf{R}^m$ be a function, let x_0 be an interior point of E, and let $v\in\mathbf{R}^n$. If f is differentiable at x_0, then f is also differentiable in the direction v at x_0, and

$$D_vf(x_0)=f'(x_0)v.$$

Exercise 4. Define $f\colon\mathbf{R}^2\to\mathbf{R}$ by $f(x,y):=x^3/(x^2+y^2)$ when $(x,y)\neq(0,0)$, and $f(0,0):=0$. Show that for any $v\in\mathbf{R}^2$, f is differentiable at $(0,0)$ in the direction v. However, show that f is not differentiable at $(0,0)$.

Exercise 5. Prove the following statements.

- Let $L\colon\mathbf{R}^n\to\mathbf{R}^m$ be a linear transformation. Show that there exists a real number $M>0$ such that $\|Lx\|\leq M\|x\|$, for all $x\in\mathbf{R}^n$. (Hint: first, using the equivalence between linear transformations and matrices, write L in terms of a matrix A. Then, set M to be equal to the sum of the absolute values of the entries of A. Use the triangle inequality a lot. There are many different ways to do this exercise, some of which use a different value of M. For example, you could try using the Cauchy-Schwarz inequality.) In particular, conclude that any linear transformation $L\colon\mathbf{R}^n\to\mathbf{R}^m$ is continuous.
- Let E be a subset of $\mathbf{R}^n$. Assume that $f\colon E\to\mathbf{R}^m$ is differentiable at an interior point x_0 of E. Then f is also continuous at x_0.
- (**The Chain Rule in Multiple Variables**) Let E be a subset of $\mathbf{R}^n$, let F be a subset of $\mathbf{R}^m$, let $f\colon E\to F$ be a function, and let $g\colon F\to\mathbf{R}^p$. Let x_0 be a point in the interior of E. Assume that f is differentiable at x_0 and that $f(x_0)$ is in the

interior of F. Assume also that g is differentiable at $f(x_0)$. Show that $g \circ f \colon E \to \mathbf{R}^p$ is also differentiable at x_0, and

$$(g \circ f)'(x_0) = g'(f(x_0))f'(x_0).$$

(Hint: it may be helpful to review the proof of the single variable chain rule. It is probably easiest to use the sequence definition of a limit.)

Exercise 6. Let $f \colon \mathbf{R}^n \to \mathbf{R}$ be a differentiable function, and let $\alpha \in \mathbf{R}$. Suppose f is homogeneous of degree α. That is, for all $x \in \mathbf{R}^n$ and for all $t > 0$, we have $f(tx) = t^{\alpha} f(x)$. Prove that $f'(x)x = \alpha f(x)$, for all $x \in \mathbf{R}^n$.

Exercise 7. Define $f \colon \mathbf{R}^2 \to \mathbf{R}$ by $f(x, y) := (x^3 y)/(x^2 + y^2)$ when $(x, y) \neq (0, 0)$, and $f(0, 0) := 0$. Show that f is continuously differentiable, and the double derivatives $\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} f$ and $\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1}$ exist, but these derivatives are not equal at $(0, 0)$.