Please provide complete and well-written solutions to the following exercises.

Due January 15, in the discussion section.

Assignment 1

Exercise 1. Let $(X, \|\cdot\|)$ be a normed linear space. Define $d\colon X \times X \to \mathbf{R}$ by $d(x, y) := \|x - y\|$. Show that (X, d) is a metric space.

Exercise 2. Let n be a positive integer and let $x \in \mathbf{R}^n$. Show that $\|x\|_{\ell_\infty} = \lim_{p\to\infty} \|x\|_{\ell_p}$.

Exercise 3. Let $(X, \langle\cdot,\cdot\rangle)$ be a real inner product space. Define $\|\cdot\| : X \to [0, \infty)$ by $\|x\| := \sqrt{\langle x, x\rangle}$. Show that $(X, \|\cdot\|)$ is a normed linear space. Consequently, from Exercise 1, if we define $d\colon X \times X \to [0, \infty)$ by $d(x, y) := \sqrt{\langle (x - y), (x - y)\rangle}$, then (X, d) is a metric space.

Exercise 4 (Cauchy-Schwarz Inequality). Let $(X, \langle\cdot,\cdot\rangle)$ be a complex inner product space. Let $x, y \in X$. Prove the Cauchy-Schwarz inequality for complex inner product spaces:

$$|\langle x, y\rangle| \leq \langle x, x\rangle^{1/2}\langle y, y\rangle^{1/2}.$$

Exercise 5. Consider the set A of all (x, y) in the plane $\mathbf{R}^2$ such that $x > 0$. Find the set of all adherent points of A, then find whether or not A is open or closed (or both, or neither).

Exercise 6. Let n be a positive integer. Let $x \in \mathbf{R}^n$. Let $(x^{(j)})_{j=k}^{\infty}$ be a sequence of elements of $\mathbf{R}^n$. We write $x^{(j)} = (x_1^{(j)}, \ldots, x_n^{(j)})$, so that for each $1 \leq i \leq n$, we have $x_i^{(j)} \in \mathbf{R}$, that is, $x_i^{(j)}$ is the i^{th} coordinate of $x^{(j)}$. Prove that the following three statements are equivalent.

- $(x^{(j)})_{j=k}^{\infty}$ converges to x with respect to d_{ℓ_1}.
- $(x^{(j)})_{j=k}^{\infty}$ converges to x with respect to d_{ℓ_2}.
- $(x^{(j)})_{j=k}^{\infty}$ converges to x with respect to d_{ℓ_∞}.

Exercise 7. Let (X, d) be a metric space, let E be a subset of X, and let x_0 be a point in X. Prove that the following statements are equivalent.

- x_0 is an adherent point of E.
- x_0 is either an interior point of E or a boundary point of E.
- There exists a sequence $(x_n)_{n=1}^{\infty}$ of elements of E which converges to x_0 with respect to the metric d.

Exercise 8. Let (X, d) be a metric space. Prove the following statements.

(i) Let E be a subset of X. Then E is open if and only if $E = \mathrm{int}(E)$. That is, E is open if and only if, for every $x \in E$, there exists an $r > 0$ such that $B(x, r) \subseteq E$.

(ii) Let E be a subset of X. Then E is closed if and only if E contains all of its adherent points, i.e. when $E = \overline{E}$. That is, E is closed if and only if, for every convergent sequence $(x_n)_{n=0}^{\infty}$ consisting of elements of E, the limit $\lim_{n\to\infty} x_n$ of the sequence also lies in E.

(iii) For any $x_0 \in X$, for any $r > 0$, the open ball $B(x_0, r)$ is an open set. The set $\{x \in X : d(x, x_0) \leq r\}$ is a closed set. The latter set is sometimes called the **closed ball** of radius r centered at x_0.

(iv) Let $x_0 \in X$. Then the singleton set $\{x_0\}$ is closed.

(v) If E is a subset of X, then E is open if and only if $X \smallsetminus E$ is closed. Here we have denoted $X \smallsetminus E := \{x \in X : x \notin E\}$ as the complement of E in X.

(vi) If $E_1, \ldots, E_n$ is a finite collection of open sets, then $E_1 \cap \cdots \cap E_n$ is an open set. If $F_1, \ldots, F_n$ is a finite collection of closed sets, then $F_1 \cup \cdots \cup F_n$ is a closed set.

(vii) If $\{E_\alpha\}_{\alpha\in I}$ is collection of open sets, (where the index set I can be finite, countable, or uncountable), then $\cup_{\alpha\in I} E_\alpha$ is an open set. If $\{F_\alpha\}_{\alpha\in I}$ is collection of closed sets, (where the index set I can be finite, countable, or uncountable), then $\cap_{\alpha\in I} F_\alpha$ is a closed set.

(viii) If E is any subset of X, then $\mathrm{int}(E)$ is the largest open set contained in E. That is, $\mathrm{int}(E)$ is open, and if V is any open set such that $V \subseteq E$, then $V \subseteq \mathrm{int}(E)$ also. Similarly, $\overline{E}$ is the smallest closed set containing E. That is, $\overline{E}$ is closed, and if V is any closed set such that $V \supseteq E$, then $V \supseteq \overline{E}$ also.

Exercise 9. Let $(x^{(j)})_{j=k}^{\infty}$ be a sequence of elements of a metric space (X, d) which converges to some limit $x \in X$. Prove that every subsequence of $(x^{(j)})_{j=k}^{\infty}$ also converges to x.

Exercise 10. Let $(x^{(j)})_{j=k}^{\infty}$ be a sequence of elements of a metric space (X, d) which converges to some limit $x \in X$. Prove that $(x^{(j)})_{j=k}^{\infty}$ is also a Cauchy sequence.

Exercise 11. Prove the following statements.

- Let (X, d) be a metric space, and let Y be a subset of X, so that $(Y, d|_{Y\times Y})$ is a metric space. If $(Y, d|_{Y\times Y})$ is complete, then Y is closed in (X, d).
- Conversely, assume that (X, d) is a complete metric space and that Y is a closed subset of X. Then $(Y, d|_{Y\times Y})$ is complete.

Exercise 12. Let X be a subset of the real line $\mathbf{R}$ and let I be a set. The set X is said to be **open** if and only if there exists a (possibly uncountable) collection of open intervals $\{(a_\alpha, b_\alpha)\}_{\alpha\in I}$ where $a_\alpha < b_\alpha$ are real numbers for all $\alpha \in I$, so that $X = \cup_{\alpha\in I}(a_\alpha, b_\alpha)$. Assume that X is open. Conclude that there exists a set J which is either finite or countable, and there exists a disjoint collection of open intervals $\{(c_\alpha, d_\alpha)\}_{\alpha\in J}$ which is either finite or countable, where $c_\alpha < d_\alpha$ are real numbers for all $\alpha \in J$, so that $X = \cup_{\alpha\in J}(c_\alpha, d_\alpha)$. (Hint: given any $x \in X$, consider the largest open interval that contains x and that is contained in X. Consider then the set of all such intervals, for all $x \in X$.)

Remark 1. The analogous statement for $\mathbf{R}^2$ is not true.