131B Final Solutions

1. Question 1

(a) There exists a continuous function $f\colon \mathbf{R} \to \mathbf{R}$ such that, for every $x \in \mathbf{R}$, f is not differentiable at x.

Solution. TRUE. This was done in Exercise 2 of homework 5. We used $\sum_{j=1}^{\infty} 4^{-j}\cos(32^j \pi x)$.

(b) Let $f\colon \mathbf{R} \to \mathbf{R}$ be an infinitely differentiable function. Then f is equal to its Taylor series centered at the origin.

Solution. FALSE. This was done in Exercise 7 of homework 5. The example was $f(x) = e^{-1/x^2}$, with $f(0) := 0$ for $x = 0$.

(c) Let $C([0,1];\mathbf{R})$ denote the space of continuous functions from $[0,1]$ to $\mathbf{R}$, where we use the sup-norm as the metric on this space. Then $C([0,1];\mathbf{R})$ is compact.

Solution. FALSE. For any $j > 3$, $j \in \mathbf{Z}$, consider the continuous, piecewise linear function $f_j\colon [0,1] \to \mathbf{R}$ where $f_j(0) = 0$, $f_j(2^{-j}) = 0$, $f_j(2^{-j+1}) = 1$, $f_j(1) = 0$, and f_j is linear in between these points. (So, $f_j(x) = 0$ when $x \in [0, 2^{-j}]$, $f_j(x) = 2^j x - 1$ when $x \in [2^{-j}, 2^{-j+1}]$, etc.) By construction, $f_{j+1}(2^{-j}) = 1$, while $f_j(2^{-j}) = 0$, and $f_k(2^{-j}) = 0$ for all $3 \leq k \leq j$. That is, for any k with $3 \leq k \leq j$, we have $\|f_{j+1} - f_k\|_\infty \geq 1$. Therefore, the sequence of functions $(f_j)_{j=3}^{\infty}$ has no convergent subsequence. (If it had a convergent subsequence, then f_{j_k} would converge uniformly to some function f as $k \to \infty$, but then $\left\|f_{j_{k+1}} - f_{j_\ell}\right\|_\infty \geq 1$ for all $1 \leq \ell \leq k$, so for all $k > K$, we have $\|f_{j_k} - f\|_\infty < 1/3$, so that $1 \leq \left\|f_{j_{K+1}} - f_{j_K}\right\|_\infty \leq \left\|f_{j_{K+1}} - f\right\|_\infty + \|f - f_{j_K}\|_\infty < 1/3 + 1/3 = 2/3$, a contradiction.)

(d) For all $x \in \mathbf{R}$, we have $-\log(1-x) = \sum_{j=1}^{\infty} x^j/j$.

Solution. FALSE. This identity only holds when $x \in (-1,1)$. For example, when $x = -2$, the left side is defined, but the right side diverges.

(e) Let $T(x)$ denote the Taylor series of $\sin(x)$ at the origin. Then the function $f\colon \mathbf{R} \to \mathbf{R}$ defined by $f(x) = \sin(x)$ satisfies $f(x) = T(x)$ for all $x \in \mathbf{R}$.

Solution. TRUE. The sin function is defined to be its own Taylor series at the origin. Also, e.g. by the ratio test, this series has radius of convergence $R = +\infty$.

(f) Let n, m be positive integers. Then every linear transformation $f\colon \mathbf{R}^n \to \mathbf{R}^m$ is continuous.

Solution. TRUE. We showed this in Homework 8, Exercise 5.

(g) Let $(x,y) \in \mathbf{R}^2$, and define $f\colon \mathbf{R}^2 \to \mathbf{R}^2$ by

$$f(x,y) := \left(x + y^3, x + \frac{x}{x^2+y^2}\right).$$

Then $\lim_{(x,y)\to(0,0)} f(x,y)$ exists.

Solution. FALSE. Consider $(a_j, b_j) = (1/j, 0)$, $j > 1$. Then $f(a_j,b_j) = (j^{-1}, j^{-1} + j)$, so $\lim_{j\to\infty} f(a_j,b_j)$ does not exist. Since $(a_j,b_j) \to (0,0)$ as $j \to \infty$, we conclude that $\lim{(x,y) \to (0,0)} f(x,y)$ does not exist.

2. Question 2

Let n be a positive integer. Let $(\mathbf{R}^n, d_{\ell_2})$ denote the Euclidean space $\mathbf{R}^n$ with the usual Euclidean metric d_{ℓ_2}. Prove that $(\mathbf{R}^n, d_{\ell_2})$ is a metric space. (Hint: you may freely use the Cauchy-Schwarz inequality.)

Solution. This follows from Exercise 3 on Homework 1. We recall the argument. We need to show that $d(x,z) \leq d(x,y) + d(y,z)$ for all $x,y,z \in \mathbf{R}^n$ (denoting $d = d_{\ell_2}$). We prove the stronger statement $\|a+b\|_{\ell_2} \leq \|a\|_{\ell_2} + \|b\|_{\ell_2}$. This is equivalent to showing its square: $\langle a+b, a+b\rangle \leq \langle a,a\rangle + 2\|a\|_{\ell_2}\|b\|_{\ell_2} + \langle b,b\rangle$. That is, it suffices to show that

$$\langle a,a\rangle + \langle b,b\rangle + 2\langle a,b\rangle \leq \langle a,a\rangle + 2\|a\|_{\ell_2}\|b\|_{\ell_2} + \langle b,b\rangle.$$

That is, it suffices to show that

$$2\langle a,b\rangle \leq 2\|a\|_{\ell_2}\|b\|_{\ell_2}.$$

This is exactly the Cauchy-Schwarz inequality.

3. Question 3

Describe the set of all complex numbers $z \in \mathbf{C}$ such that $\sum_{j=0}^{\infty} z^j/j^2$ converges.

Solution. From the ratio test, we see that $\frac{|z|^{j+1}\left|1/(j+1)^2\right|}{|z|^j|1/j^2|} = |z|\frac{j^2}{(j+1)^2} \to |z|$ as $j \to \infty$. So, if $|z| < 1$, then $\sum_{j=0}^{\infty} z^j/j^2$ converges, by the ratio test. And if $|z| > 1$, then $\sum_{j=0}^{\infty} z^j/j^2$ diverges, by the ratio test. The only remaining points to check occur when $|z| = 1$. In this case, we have

$$\left|\sum_{j=0}^{\infty} z^j/j^2\right| \leq \sum_{j=0}^{\infty} |z|^j/j^2 = \sum_{j=0}^{\infty} 1/j^2 < \infty.$$

That is, the sum is absolutely convergent when $|z| = 1$. So, the sum converges when $|z| = 1$. In conclusion, the sum converges if and only if $|z| \leq 1$.

4. Question 4

Let $x \in \mathbf{R}$, and let j be a positive integer. Define the function

$$f_j(x) := \frac{x}{1+jx^2}.$$

(a) Show that the sequence of functions $(f_j)_{j=1}^{\infty}$ converges uniformly to a function f.

Solution. Let $f(x) = 0$ for all $x \in \mathbf{R}$. Let $j > 0$, $j \in \mathbf{Z}$. Let $h_j(x) = 1/(1+jx^2)$ for any $j > 0$, $j \in \mathbf{Z}$. Note that $\lim_{x\to\infty} h_j(x) = 0 = \lim_{x\to-\infty} h_j(x)$. Also, $h_j'(x) = -2jx/(1+jx^2)$. That is, on the set $(-\infty, -j^{-1/4}] \cup [j^{1/4}, +\infty)$, h_j achieves its maximum value at $x = j^{-1/4}$ and at $x = -j^{-1/4}$. This maximum value is $h_j(j^{-1/4}) = 1/(1+j^{1/2})$.

For any $x \in [-j^{-1/4}, j^{1/4}]$, we use the bound $|f_j(x)| \leq |x| \leq j^{-1/4}$, and for any other $x \in \mathbf{R}$, we use the bound $|f_j|(x) \leq 1/(1+j^{1/2})$. That is, for any $x \in \mathbf{R}$, we have $|f_j(x)| \leq \max(j^{-1/4}, 1/(1+j^{1/2}))$. That is, for any $j > 0$, we have $d_\infty(f, f_j) \leq \max(j^{-1/4}, 1/(1+j^{1/2}))$. That is, f_j converges to f uniformly as $j \to \infty$.

(b) Show that, if $x \neq 0$, then $f'(x) = \lim_{j\to\infty} f_j'(x)$. Show that, if $x = 0$, then $f'(x) \neq \lim_{j\to\infty} f_j'(x)$.

Note: $f_j'(x) = \frac{1+jx^2-x(2jx)}{(1+jx^2)^2} = \frac{1-jx^2}{(1+jx^2)^2}$. So, if $x \neq 0$, then $\lim_{j\to\infty} f_j'(x) = \lim_{j\to\infty} \frac{-jx^2}{(1+jx^2)^2} = \lim_{j\to\infty} \frac{-jx^2}{1+2jx^2+j^2x^4} = 0$, since the numerator has a factor of j, but the denominator has a factor of j^2 (since $x \neq 0$). Since $f = 0$, we have $f'(x) = 0$, so $f'(x) = \lim_{j\to\infty} f_j'(x)$. If $x = 0$, then $f_j'(x) = 1$ for all $j > 1$, while $f'(x) = 0$, so $f'(x) \neq \lim_{j\to\infty} f_j'(x)$.

5. Question 5

Let $f\colon \mathbf{R}\to\mathbf{R}$ and let $g\colon \mathbf{R}\to\mathbf{R}$ be continuous functions. Suppose f is nonzero only in the interval $[0,1]$, and suppose g is constant in the interval $[0,2]$. That is, $f(x)=0$ for all $x\in(-\infty,0)\cup(1,\infty)$ and there exists $c\in\mathbf{R}$ such that $g(x)=c$ for all $x\in[0,2]$. Show that the convolution $f*g$ is constant on the interval $[1,2]$. (Here we define $f*g(t)=\int_{-\infty}^{\infty}f(y)g(t-y)dy$, for any $t\in\mathbf{R}$.)

Solution. Note that $f*g(t)=\int_{-\infty}^{\infty}f(y)g(t-y)dy=\int_0^1 f(y)g(t-y)dy$, since f is nonzero only on the interval $[0,1]$. Let $t\in[1,2]$, and let $y\in[0,1]$. Then $(t-y)\in[0,2]$. So, by assumption on g, we have $g(t-y)=c$ whenever $t\in[1,2]$ and $y\in[0,1]$. That is, for any $t\in[1,2]$, we have $f*g(t)=\int_0^1 f(y)cdy$. Since $\int_0^1 f(y)cdy$ does not depend on t, we conclude that $f*g(t)$ is constant for all $t\in[1,2]$.

6. Question 6

Let $f\colon[0,1]\to\mathbf{R}$ be a continuous $\mathbf{Z}$-periodic function. Assume that, for all positive integers n, we have $\int_0^1 f(x)x^n dx=0$. Conclude that $f(x)=0$ for all $x\in[0,1]$. (Hint: first show that $\int_0^1 f(x)P(x)dx=0$ for any polynomial P. Then, use the Weierstrass approximation theorem to show that $\int_0^1 f(x)f(x)dx=0$.) (Don't use the Weierstrass approximation for trigonometric polynomials.)

Solution. Let P be a polynomial. Then there exists a positive integer n and constants $a_n,\ldots,a_0$ such that $P(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x_1+a_0$. That is, $\int_0^1 f(x)P(x)dx=\sum_{i=0}^n a_i\int_0^1 f(x)x^i dx=0$, since the final term is a sum of zeros, by assumption. (Recall that $|f|$, $f\cdot f$, $(f-P)$ and all of their products are all continuous functions, so they are all Riemann integrable.) Now, let $\varepsilon>0$. Let P be a polynomial such that $|f(x)-P(x)|<\varepsilon$ for all $x\in[0,1]$. Then

$$\left|\int_0^1 f(x)f(x)dx-\int_0^1 f(x)P(x)dx\right|=\left|\int_0^1 f(x)(f(x)-P(x))dx\right|$$
$$\leq\int_0^1 |f(x)|\,|f(x)-P(x)|<\varepsilon\int_0^1|f(x)|\,dx.$$

As we just showed, $\int_0^1 f(x)P(x)dx=0$. That is, we have shown that $|\int_0^1 f(x)f(x)dx|<\varepsilon\int_0^1|f(x)|\,dx$. Since $\varepsilon>0$ is arbitrary, we conclude that $\int_0^1|f(x)|^2\,dx=0$. Since f is continuous, we conclude that f is zero as well. (If f were nonzero, there would exist some x and $\delta>0$ such that $|f(x)|>\delta$. By continuity, there would then exist some $\eta>0$ such that $|f(y)|>\delta/2$ for all $y\in(x-\eta,x+\eta)$. So, $\int_0^1|f(t)|^2\,dt\geq\int_{x-\eta}^{x+\eta}|f(t)|^2\,dt>\eta\delta>0$, a contradiction.)

7. Question 7

Let $(x,y,z)\in\mathbf{R}^3$. Define $f\colon\mathbf{R}^3\to\mathbf{R}^3$ by

$$f(x,y,z):=(x^2,xyz,z^2+xy^2).$$

Prove that f is differentiable. Then, compute the differential of f.

Solution. Each component of f is continuously differentiable. So, by Theorem 4.7 in the final set of notes, f is differentiable. Moreover, by Theorem 4.7, the differential is given by the formula

$$f'(x,y,z)(v_1,v_2,v_3) = \sum_{j=1}^{3} v_j \frac{\partial f}{\partial x_j}(x,y,z) = v_1(2x, yz, y^2) + v_2(0, xz, 2xy) + v_3(0, yz, 2z).$$

Here $(v_1, v_2, v_3) \in \mathbf{R}^3$.

8. Question 8

Let $f\colon \mathbf{R} \to (0,\infty)$ be a positive, real analytic function such that $f'(x) = f(x)$ for all $x \in \mathbf{R}$. Show that there exists a real number $C \in \mathbf{R}$ such that $f(x) = Ce^x$ for all $x \in \mathbf{R}$. (Hint: there are at least three ways to prove this. One proof uses the logarithm function, another proof uses the function e^{-x}, and a third proof uses power series. You only need to provide one proof.)

Solution 1. Since f is positive, the function $h(x) = \log f(x)$ is well-defined. Since f is analytic, it is differentiable, so by the chain rule we have $h'(x) = f'(x)/f(x) = f(x)/f(x) = 1$ (using our assumption $f'(x) = f(x)$ for all $x \in \mathbf{R}$). So, by the Fundamental Theorem of Calculus, there exists $c \in \mathbf{R}$ with $h(x) = x + c$ for all $x \in \mathbf{R}$. That is, $\log f(x) = x + c$ for all $x \in \mathbf{R}$. Exponentiating both sides, we have $f(x) = e^{x+c} = (e^c)e^x$ for all $x \in \mathbf{R}$. So, define $C = e^c$.

Solution 2. Define $h(x) = e^{-x}f(x)$. Note that e^{-x} and $f(x)$ are both analytic, so h is analytic as well, by Theorem 8.22 in the third set of notes. In particular, h is differentiable. And by the product rule, $h'(x) = e^{-x}f'(x) - e^{-x}f(x) = e^{-x}(f'(x) - f(x)) = 0$ for all $x \in \mathbf{R}$ (using our assumption $f'(x) = f(x)$ for all $x \in \mathbf{R}$). So, by the Fundamental Theorem of Calculus, there exists $C \in \mathbf{R}$ with $h(x) = C$ for all $x \in \mathbf{R}$. That is, $e^{-x}f(x) = C$, so that $f(x) = Ce^x$ for all $x \in \mathbf{R}$.

Solution 3. Since f is real analytic on $\mathbf{R}$, we can write f as its Taylor series $f(x) = \sum_{j=0}^{\infty} a_j x^j$ (Corollary 8.15 in the second set of notes). Since f is once again real analytic, we can differentiate it term by term to get $f'(x) = \sum_{j=0}^{\infty} a_j j x^{j-1}$ (Theorem 8.5 in the second set of notes). Since $f'(x) = f(x)$, we conclude by the Uniqueness of power series that $a_j = (j+1)a_{j+1}$ for all $j \geq 0$ (Corollary 8.19 in the second set of notes). For example, $a_1 = 2a_2$, $a_3 = 3a_3$, and so on. We prove by induction that $a_j = a_0/j!$. Since $0! = 1$, the base case holds. We therefore induct on j. Assume $a_j = a_0/j!$. We then prove $a_{j+1} = a_0/(j+1)!$. Since $a_{j+1} = a_j/(j+1)$, the inductive hypothesis says that $a_{j+1} = a_0/((j!)(j+1)) = a_0/(j+1)!$. The induction is therefore complete. We have shown that $a_j = a_0/j!$. That is, $f(x) = a_0 \sum_{j=0}^{\infty} x^j/j! = a_0 e^x$. So, set $C = a_0$.

9. Question 9

Let $f\colon [0,1] \to \mathbf{R}^3$ be a continuous function such that $f(0) = (0,0,0)$ and $f(1) = (1,1,1)$. Let S denote the subset of $\mathbf{R}^3$ defined by

$$S = \{(x,y,z) \in \mathbf{R}^3 \colon x^2 + y^2 + z^2 = 1\}.$$

That is, S is the unit sphere in $\mathbf{R}^3$. Prove that there exists $t \in [0,1]$ and there exists $s \in S$ such that $f(t) = s$. (Hint: how did we prove the intermediate value theorem?)

Solution. We argue by contradiction. Suppose no such t exists. Let B denote the open ball where $\{(x,y,z)\in\mathbf{R}^3\colon x^2+y^2+z^2<1\}$, and let D denote the set $\{(x,y,z)\in\mathbf{R}^3\colon x^2+y^2+z^2>1\}$. Note that B and D are both open in $\mathbf{R}^3$. Since no such t exists, we conclude that $B\cap f([0,1])$ and $D\cap f([0,1])$ are both relatively open with respect to $f([0,1])$. That is, $f([0,1])$ is disconnected. (Here we also used that $B\cap f([0,1])$ and $D\cap f([0,1])$ are nonempty, which follows since $f(0)=(0,0,0)\in B$ and $f(1)=(1,1,1)\in D$.) However, since $f([0,1])$ is disconnected, we have found a contradiction. We know $f([0,1])$ is connected since f is continuous and $[0,1]$ is connected (Theorem 8.6 from the first set of notes). In conclusion, such a t must exist.

10. Question 10

Let $f\colon[0,1]\to\mathbf{C}$ and let $g\colon[0,1]\to\mathbf{C}$ be continuously differentiable, $\mathbf{Z}$-periodic functions. For any $n\in\mathbf{Z}$, define $a_n=\int_0^1 f(x)e^{-2\pi inx}dx$ and define $b_n=\int_0^1 g(x)e^{-2\pi inx}dx$. Prove that

$$\int_0^1 f(x)\overline{g(x)}dx=\lim_{N\to\infty}\sum_{n=-N}^{N}a_n\overline{b_n}.$$

(Hint: in the case $f=g$, this is exactly Plancherel's Theorem. So, maybe try to mimic the proof of Plancherel's Theorem. That is, first try to prove the statement when f and g are trigonometric polynomials.)

Solution. For any $n\in\mathbf{Z}$ and $x\in\mathbf{R}$, let $e_n(x)=e^{2\pi inx}$. Let f and g be trigonometric polynomials. That is, assume there exists $N\in\mathbf{N}$ such that $f=\sum_{n=-N}^{N}a_ne_n$ and $g=\sum_{m=-N}^{N}b_me_m$. Then

$$\langle f,g\rangle=\left\langle\sum_{n=-N}^{N}a_ne_n,\sum_{m=-N}^{N}b_me_m\right\rangle=\sum_{n,m=-N}^{N}a_n\overline{b_m}\langle e_n,e_m\rangle=\sum_{n=-N}^{N}a_n\overline{b_m}.$$

Here we used that $\langle e_n,e_m\rangle=0$, unless $n=m$, in which case $\langle e_n,e_m\rangle=1$ (Homework 7, Exercise 6). That is, we have proven the required statement for trigonometric polynomials.

Now, let f,g be any continuous $\mathbf{Z}$-periodic functions. Let $\varepsilon>0$. Let $f_N=\sum_{n=-N}^{N}a_ne_n$ and let $g_N=\sum_{n=-N}^{N}b_ne_n$. By Fourier inversion (Theorem 7.1 in the third set of notes), there exists $N>0$ such that $\|f-f_N\|_2<\varepsilon$ and such that $\|g-g_N\|_2<\varepsilon$. Then

$$\begin{aligned}\left|\langle f,g\rangle-\sum_{n=-N}^{N}a_n\overline{b_n}\right|&=|\langle f,g\rangle-\langle f_N,g_N\rangle| &&\text{, by what we proved already}\\&=|\langle f-f_N,g\rangle+\langle f_N,g-g_N\rangle|\leq|\langle f-f_N,g\rangle|+|\langle f_N,g-g_N\rangle|\\&\leq\|f-f_N\|_2\|g\|_2+\|f_N\|_2\|g-g_N\|_2 &&\text{, by the Cauchy-Schwarz inequality.}\end{aligned}$$

Now, $\|f_N\|_2\leq\|f\|_2$ for all $N>0$ by Plancherel's Theorem (Theorem 7.4 in the third set of notes). In conclusion, we have shown that: for all $\varepsilon>0$, there exists $N>0$ such that

$$\left|\langle f,g\rangle-\sum_{n=-N}^{N}a_n\overline{b_n}\right|\leq\varepsilon\|g\|_2+\varepsilon\|f\|_2.$$

That is, $\lim_{N\to\infty}\sum_{n=-N}^{N}a_n\overline{b_n}=\langle f,g\rangle$, as desired.

11. Question 11

Let $f\colon [0,1] \to \mathbf{R}$ be a continuously differentiable, $\mathbf{Z}$-periodic function. Assume also that $\int_0^1 f = 0$. Prove that

$$\int_0^1 (f'(x))^2 dx \geq \int_0^1 (f(x))^2 dx.$$

Solution. Suppose f has Fourier expansion $\sum_{n\in\mathbf{Z}} a_n e^{2\pi inx}$ and f' has Fourier expansion $\sum_{n\in\mathbf{Z}} b_n e^{2\pi inx}$. Note that $\int_0^1 f'(x)e^{-2\pi inx} = 2\pi in \int_0^1 f(x)e^{-2\pi inx}$, by integrating by parts. That is, $b_n = 2\pi i n a_n$. Using Plancherel's Theorem (Theorem 7.4 in the third set of notes), we are required to show that

$$\sum_{n\in\mathbf{Z}} |b_n|^2 \geq \sum_{n\in\mathbf{Z}} |a_n|^2 .$$

We in fact show the stronger inequality $|b_n| \geq |a_n|$, for all $n \in \mathbf{Z}$. Since $b_n = 2\pi i n a_n$, we automatically have $|b_n| = |2\pi n| \, |a_n| \geq |a_n|$, as long as $n \neq 0$. In the case that $n = 0$, we have $a_0 = \int_0^1 f(x)dx = 0$ by assumption, so therefore $|b_0| \geq 0 = |a_0|$. In conclusion, $|b_n| \geq |a_n|$ for all $n \in \mathbf{Z}$, as desired.