Analysis 2, Winter 2015, UCLA Instructor: Steven Heilman

Name: ______________________ UCLA ID: ______________ Date: ___________

Signature: ______________________.
(By signing here, I certify that I have taken this test while refraining from cheating.)

Final Exam

This exam contains 17 pages (including this cover page) and 11 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam.
You are required to show your work on each problem on this exam.
The following rules apply:

- You have 180 minutes to complete the exam.
- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Do not write in the table to the right. Good luck!

Problem	Points	Score
1	20	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
Total:	120	

Reference sheet

Below are some definitions that may be relevant.

A **metric space** (X, d) is a set X together with a function $d\colon X \times X \to [0, \infty)$ which satisfies the following properties. (i) For all $x \in X$, we have $d(x, x) = 0$. (ii) For all $x, y \in X$ with $x \neq y$, we have $d(x, y) > 0$. (Positivity) (iii) For all $x, y \in X$, we have $d(x, y) = d(y, x)$. (Symmetry) (iv) For all $x, y, z \in X$, we have $d(x, z) \leq d(x, y) + d(y, z)$. (Triangle inequality)

Let X be a vector space over $\mathbf{R}$. A **normed linear space** $(X, \|\cdot\|)$ is a vector space X over $\mathbf{R}$ together with a norm function $\|\cdot\| : X \to [0, \infty)$ which satisfies the following properties. (i) $\|0\| = 0$. (ii) For all $x \in X$ with $x \neq 0$, we have $\|x\| > 0$. (Positivity) (iii) For all $x \in X$ and for all $\alpha \in \mathbf{R}$, we $\|\alpha x\| = |\alpha| \, \|x\|$. (Homogeneity) (iv) For all $x, y \in X$, we have $\|x + y\| \leq \|x\| + \|y\|$. (Triangle inequality)

Let X be a vector space over $\mathbf{R}$. A **real inner product space** $(X, \langle \cdot, \cdot \rangle)$ is a vector space X over $\mathbf{R}$ together with an inner product function $\langle \cdot, \cdot \rangle \colon X \times X \to \mathbf{R}$ which satisfies the following properties. (i) $\langle 0, 0 \rangle = 0$. (ii) For all $x \in X$ with $x \neq 0$, we have $\langle x, x \rangle > 0$. (iii) For all $x, y \in X$, we have $\langle x, y \rangle = \langle y, x \rangle$. (Symmetry) (iv) For all $x \in X$ and for all $\alpha \in \mathbf{R}$, we $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$. (Homogeneity) (v) For all $x, y, z \in X$, we have $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$. (Linearity)

Let (X, d) be a metric space. We say that (X, d) is **complete** if and only if the following property holds. For any Cauchy sequence $(x^{(j)})_{j=k}^{\infty}$ of elements of X, then there exists some $x \in X$ such that $(x^{(j)})_{j=k}^{\infty}$ converges to x with respect to d.

A metric space (X, d) is said to be **compact** if and only if every sequence in (X, d) has at least one convergent subsequence. We say that $Y \subseteq X$ is **compact** if and only if the metric space $(Y, d|_{Y \times Y})$ is compact.

Let (X, d) be a metric space. We say that X is **disconnected** if and only if there exist disjoint open sets V, W in X such that $V \cup W = X$. (Equivalently, X is disconnected if and only if X contains a proper non-empty subset which is both open and closed.) We say that X is **connected** if and only if X is not disconnected. We say that $Y \subseteq X$ is **connected** if and only if the metric space $(Y, d|_{Y \times Y})$ is connected. We say that Y is **disconnected** if and only if the metric space $(Y, d|_{Y \times Y})$ is disconnected.

Let (X, d_X) and (Y, d_Y) be metric spaces. Let $(f_j)_{j=1}^{\infty}$ be a sequence of functions from X to Y. Let $f\colon X \to Y$ be another function. We say that $(f_j)_{j=1}^{\infty}$ **converges pointwise** to f on X if and only if, for every $x \in X$, we have

$$\lim_{j \to \infty} f_j(x) = f(x).$$

That is, for all $x \in X$, we have

$$\lim_{j \to \infty} d_Y(f_j(x), f(x)) = 0.$$

That is, for every $x \in X$ and for every $\varepsilon > 0$, there exists $J > 0$ such that, for all $j > J$, we have $d_Y(f_j(x), f(x)) = 0$.

Let (X, d_X) and (Y, d_Y) be metric spaces. Let $(f_j)_{j=1}^{\infty}$ be a sequence of functions from X to Y. Let $f\colon X \to Y$ be another function. We say that $(f_j)_{j=1}^{\infty}$ **converges uniformly** to f on X if and only if, for every $\varepsilon > 0$, there exists $J > 0$ such that, for all $j > J$ and for all $x \in X$ we have $d_Y(f_j(x), f(x)) = 0$.

Let $a \in \mathbf{R}$ and let $r > 0$. Let E be a subset of $\mathbf{R}$ such that $(a-r, a+r) \subseteq E$. Let $f\colon E \to \mathbf{R}$. We say that the function f is **real analytic on** $(a-r, a+r)$ if and only if there exists a power series $\sum_{j=0}^{\infty} a_j(x-a)^j$ centered at a with radius of convergence R such that $R \geq r$ and such that this power series converges to f on $(a-r, a+r)$.

A function $f\colon \mathbf{R} \to \mathbf{C}$ is **Z-periodic** if and only if $f(x+k) = f(x)$ for all $x \in \mathbf{R}$ and for all $k \in \mathbf{Z}$. The space of all complex-valued $\mathbf{Z}$-periodic functions is denoted by $C(\mathbf{R}/\mathbf{Z};\mathbf{C})$.

For any function $f \in C(\mathbf{R}/\mathbf{Z};\mathbf{C})$ and any integer $n \in \mathbf{Z}$, we define the n^{th} **Fourier coefficient** of f, denoted $\widehat{f}(n)$, by

$$\widehat{f}(n) := \langle f, e_n \rangle = \int_0^1 f(x) e^{-2\pi i n x} dx.$$

Let $f, g \in C(\mathbf{R}/\mathbf{Z};\mathbf{C})$. We define the **convolution** $f * g \in C(\mathbf{R}/\mathbf{Z};\mathbf{C})$ of f and g by the formula

$$f * g(x) := \int_0^1 f(y) g(x-y) dy, \quad \forall\, x \in \mathbf{R}.$$

Let n, m be positive integers. We say that $L\colon \mathbf{R}^n \to \mathbf{R}^m$ is a **linear transformation** if and only if (i) for all $x, y \in \mathbf{R}^n$, we have $L(x+y) = L(x) + L(y)$, and (ii) for all $x \in \mathbf{R}^n$ and for all $\alpha \in \mathbf{R}$, we have $L(\alpha x) = \alpha L(x)$.

Let E be a subset of $\mathbf{R}^n$, let $f\colon E \to \mathbf{R}^m$ be a function, let $x_0 \in E$, and let $L\colon \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation. We say that f is **differentiable at** x_0 **with derivative** L if and only if we have

$$\lim_{x \to x_0; x \in E} \frac{\|f(x) - (f(x_0) + L(x - x_0))\|}{\|x - x_0\|} = 0.$$

Here $\|x\|$ denotes the ℓ_2 norm of x:

$$\|(x_1, \ldots, x_n)\| := (x_1^2 + \cdots + x_n^2)^{1/2}.$$

1. Label the following statements as TRUE or FALSE. Briefly justify your answer. If the statement is false, provide a counterexample.

   (a) (3 points) There exists a continuous function $f\colon \mathbf{R} \to \mathbf{R}$ such that, for every $x \in \mathbf{R}$, f is not differentiable at x.

   TRUE FALSE (circle one)

   (b) (3 points) Let $f\colon \mathbf{R} \to \mathbf{R}$ be an infinitely differentiable function. Then f is equal to its Taylor series centered at the origin.

   TRUE FALSE (circle one)

   (c) (3 points) Let $C([0,1];\mathbf{R})$ denote the space of continuous functions from $[0,1]$ to $\mathbf{R}$, where we use the sup-norm metric $d(f,g) := \sup_{x\in[0,1]} |f(x) - g(x)|$, $f,g \in C([0,1];\mathbf{R})$ to make $C([0,1];\mathbf{R})$ into a metric space. Then $C([0,1];\mathbf{R})$ is compact.

   TRUE FALSE (circle one)

   (d) (2 points) For all $x \in \mathbf{R}$, we have $-\log(1-x) = \sum_{j=1}^{\infty} x^j/j$.

   TRUE FALSE (circle one)

(e) (3 points) Let $T(x)$ denote the Taylor series of $\sin(x)$ at the origin. Then the function $f\colon \mathbf{R} \to \mathbf{R}$ defined by $f(x) = \sin(x)$ satisfies $f(x) = T(x)$ for all $x \in \mathbf{R}$.

TRUE FALSE (circle one)

(f) (3 points) Let n, m be positive integers. Then every linear transformation $f\colon \mathbf{R}^n \to \mathbf{R}^m$ is continuous. (As usual, since we have not specified a metric, we mean that we are using the ℓ_2 metric on both $\mathbf{R}^n$ and on $\mathbf{R}^m$.)

TRUE FALSE (circle one)

(g) (3 points) Let $(x, y) \in \mathbf{R}^2$, and define $f\colon \mathbf{R}^2 \to \mathbf{R}^2$ by

$$f(x,y) := \left(x + y^3, x + \frac{x}{x^2 + y^2}\right).$$

Then $\lim_{(x,y)\to(0,0)} f(x, y)$ exists.

TRUE FALSE (circle one)

2. (10 points) Let n be a positive integer. Let $(\mathbf{R}^n, d_{\ell_2})$ denote the Euclidean space $\mathbf{R}^n$ with the usual Euclidean metric d_{ℓ_2}. Prove that $(\mathbf{R}^n, d_{\ell_2})$ is a metric space. (Hint: you may freely use the Cauchy-Schwarz inequality.)

3. (10 points) Describe the set of all complex numbers $z \in \mathbf{C}$ such that $\sum_{j=0}^{\infty} z^j/j^2$ converges.

4. Let $x \in \mathbf{R}$, and let j be a positive integer. Define the function

$$f_j(x) := \frac{x}{1 + jx^2}.$$

(a) (5 points) Show that the sequence of functions $(f_j)_{j=1}^{\infty}$ converges uniformly to some function f.

(b) (5 points) We use the function f from the first part of the question. Show that, if $x \neq 0$, then $f'(x) = \lim_{j\to\infty} f_j'(x)$. Show that, if $x = 0$, then $f'(x) \neq \lim_{j\to\infty} f_j'(x)$.

5. (10 points) Let $f\colon \mathbf{R} \to \mathbf{R}$ and let $g\colon \mathbf{R} \to \mathbf{R}$ be continuous functions. Suppose f is nonzero only in the interval $[0,1]$, and suppose g is constant in the interval $[0,2]$. That is, $f(x) = 0$ for all $x \in (-\infty, 0) \cup (1, \infty)$ and there exists $c \in \mathbf{R}$ such that $g(x) = c$ for all $x \in [0,2]$. Show that the convolution $f * g$ is constant on the interval $[1,2]$. (Here we define $f * g(t) = \int_{-\infty}^{\infty} f(y)g(t-y)dy$, for any $t \in \mathbf{R}$.)

6. (10 points) Let $f\colon [0,1] \to \mathbf{R}$ be a continuous $\mathbf{Z}$-periodic function. Assume that, for all positive integers n, we have $\int_0^1 f(x)x^n dx = 0$. Conclude that $f(x) = 0$ for all $x \in [0,1]$. (Hint: first show that $\int_0^1 f(x)P(x)dx = 0$ for any polynomial P. Then, use the Weierstrass approximation theorem to show that $\int_0^1 f(x)f(x)dx = 0$.) (Don't use the Weierstrass approximation for trigonometric polynomials.)

7. (10 points) Let $(x, y, z) \in \mathbf{R}^3$. Define $f\colon \mathbf{R}^3 \to \mathbf{R}^3$ by

$$f(x, y, z) := (x^2, xyz, z^2 + xy^2).$$

Prove that f is differentiable. Then, compute the differential of f.

8. (10 points) Let $f\colon \mathbf{R} \to (0, \infty)$ be a positive, real analytic function such that $f'(x) = f(x)$ for all $x \in \mathbf{R}$. Show that there exists a real number $C \in \mathbf{R}$ such that $f(x) = Ce^x$ for all $x \in \mathbf{R}$. (Hint: there are at least three ways to prove this. One proof uses the logarithm function, another proof uses the function e^{-x}, and a third proof uses power series. You only need to provide one proof.)

9. (10 points) Let $f\colon [0,1] \to \mathbf{R}^3$ be a continuous function such that $f(0) = (0,0,0)$ and $f(1) = (1,1,1)$. Let S denote the subset of $\mathbf{R}^3$ defined by

$$S = \{(x,y,z) \in \mathbf{R}^3 \colon x^2 + y^2 + z^2 = 1\}.$$

That is, S is the unit sphere in $\mathbf{R}^3$. Prove that there exists $t \in [0,1]$ and there exists $s \in S$ such that $f(t) = s$. (Hint: how did we prove the intermediate value theorem?)

10. (10 points) Let $f\colon [0,1] \to \mathbf{C}$ and let $g\colon [0,1] \to \mathbf{C}$ be continuously differentiable, $\mathbf{Z}$-periodic functions. For any $n \in \mathbf{Z}$, define $a_n = \int_0^1 f(x)e^{-2\pi inx}dx$ and define $b_n = \int_0^1 g(x)e^{-2\pi inx}dx$. Prove that

$$\int_0^1 f(x)\overline{g(x)}dx = \lim_{N\to\infty} \sum_{n=-N}^{N} a_n\overline{b_n}.$$

(Hint: in the case $f = g$, this is exactly Plancherel's Theorem. So, maybe try to mimic the proof of Plancherel's Theorem. That is, first try to prove the statement when f and g are trigonometric polynomials.)

11. (10 points) Let $f\colon [0,1] \to \mathbf{R}$ be a continuously differentiable, $\mathbf{Z}$-periodic function. Assume also that $\int_0^1 f = 0$. Prove that

$$\int_0^1 (f'(x))^2 dx \geq \int_0^1 (f(x))^2 dx.$$

(Scratch paper)

(More scratch paper)