131B Midterm 2 Solutions

1. Question 1

Let (X, d_X) be a metric space. For each positive integer j, let $f_j : X \to \mathbf{R}$ be a continuous function. (As usual, $\mathbf{R}$ denotes the real line with the standard metric $d(a, b) := |a - b|$, where $a, b \in \mathbf{R}$.) Suppose $(f_j)_{j=1}^{\infty}$ converges pointwise to a function $f : X \to \mathbf{R}$. Let $h : \mathbf{R} \to \mathbf{R}$ be a continuous function. Show that the sequence of functions $(h \circ f_j)_{j=1}^{\infty}$ converges pointwise to $h \circ f : X \to \mathbf{R}$. (As usual, $h \circ f_j(x) := h(f_j(x))$, and $h \circ f(x) := h(f(x))$ for all $j \geq 1$, for all $x \in X$.)

Solution. Fix $x \in X$, and let $\varepsilon > 0$. We need to find $J > 0$ such that, for all $j > J$, we have $|h(f_j(x)) - h(f(x))| < \varepsilon$. Since h is continuous at $f(x)$, there exists $\delta = \delta(x) > 0$ such that, if $y \in \mathbf{R}$ satisfies $|y - f(x)| < \delta$, then $|h(y) - h(f(x))| < \varepsilon$. Since $(f_j)_{j=1}^{\infty}$ converges pointwise to f, there exists $J = J(\delta, x) > 0$ such that, for all $j > J$, we have $|f_j(x) - f(x)| < \delta$. Using $y = f_j(x)$ in the definition of continuity of h then shows that $|h(f_j(x)) - h(f(x))| < \varepsilon$, as desired.

2. Question 2

Let $C([0, 1]; \mathbf{R})$ denote the set of continuous functions with domain $[0, 1]$ and range $\mathbf{R}$. As usual, we consider $C([0, 1]; \mathbf{R})$ to be a metric space with the metric $d(f, g) := \sup_{x \in [0,1]} |f(x) - g(x)|$, where $f, g \in C([0, 1]; \mathbf{R})$. Let V denote the subset of $C([0, 1]; \mathbf{R})$ consisting of all functions $f : [0, 1] \to \mathbf{R}$ such that $f(0) = 4f(1)$. Is V a complete subset of $C([0, 1]; \mathbf{R})$? Prove your assertion.

Solution. V is a complete subset of $C([0, 1]; \mathbf{R})$. To see this, let $(f_j)_{j=1}^{\infty}$ be a Cauchy sequence in $C([0, 1]; \mathbf{R})$. From Theorem 3.12 in the second set of notes, $C([0, 1]; \mathbf{R})$ is itself a complete metric space (since $\mathbf{R}$ is itself complete). So, $(f_j)_{j=1}^{\infty}$ converges to some function $f \in C([0, 1]; \mathbf{R})$. From Proposition 3.9 in the second set of notes, $(f_j)_{j=1}^{\infty}$ converges uniformly to f on $[0, 1]$. In particular, for any $\varepsilon > 0$, there exists $J = J(\varepsilon) > 0$ such that, for all $j > J$, we have $|f_j(0) - f(0)| < \varepsilon/2$ and $|f_j(1) - f(1)| < \varepsilon/2$. Since $f_j(0) = 4f_j(1)$ for all $j \geq 1$, we have by the triangle inequality

$$|f(0) - 4f(1)| \leq |f(0) - f_j(0)| + |f_j(0) - 4f_j(1)| + |4f_j(1) - 4f_1| \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $f(0) = 4f(1)$. That is, $f \in V$, so V is complete.

3. Question 3

Find a power series centered at the origin for the function $\tan^{-1} : \mathbf{R} \to (-\pi/2/\pi/2)$. Indicate the radius of convergence of this power series and justify your reasoning. (You may assume that $\tan^{-1}$ is differentiable, and that $\frac{d}{dx} \tan^{-1}(x) = \frac{1}{1+x^2}$.)

Solution. We have $(d/dx) \tan^{-1}(x) = 1/(1 + x^2)$. Recall that $1/(1 - x)$ has power series $\sum_{j=0}^{\infty} x^j$, so $1/(1 + x^2)$ has power series $\sum_{j=0}^{\infty} (-x^2)^j = \sum_{j=0}^{\infty} (-1)^j x^{2j}$. By inspection, this power series has radius of convergence $R = 1$, since $\limsup_{j \to \infty} |(-1)^j|^{1/(2j)} = 1$, so $R = 1/1 = 1$. So, for any $x \in (-1, 1)$, we can integrate this power series term by term to get the power series for $\tan^{-1}(x)$, using Theorem 8.5(e) in the second set of notes. That is, $\tan^{-1}(x) = \int_0^x (d/dt) \tan^{-1}(t) dt$ by the Fundamental Theorem of Calculus, so $\tan^{-1}$ has power series $\sum_{j=0}^{\infty}(-1)^j x^{2j+1}/(2j+1)$. Note that $\limsup_{j\to\infty} |(-1)^j/(2j+1)|^{1/j} = \limsup_{j\to\infty}(2j+1)^{-1/j} = 1$, so this power series has radius of convergence $R = 1/1 = 1$.

4. Question 4

Give an example of a function $f\colon \mathbf{R} \to \mathbf{R}$ such that f is not the zero function, f is infinitely differentiable, such that $f(0) = 0$ and such that $f^{(k)}(0) = 0$ for all integers $k \geq 1$. Prove that your function f satisfies these properties.

Solution. We can use the function $f(x) = e^{-1/x^2}$ for $x \neq 0$ and $f(0) := 0$. This function was shown to satisfy the required properties in Homework 5, Exercise 7.

5. Question 5

Define $f\colon \mathbf{R} \to \mathbf{R}$ by $f(x) := |x-1|$ for all $x \in \mathbf{R}$. What is the Taylor series of f at $x = 0$? What is the radius of convergence of this Taylor series? Where does this Taylor series agree with f? Does there exist any Taylor series that agrees with f on all of $\mathbf{R}$? Justify your answers.

Solution. At $x = 0$, we have $f(0) = 1$, $f'(0) = -1$, and $f^{(k)}(0) = 0$ for all $k > 1$. So, the Taylor series for f at $x = 0$ is $1 - x$. Since this Taylor series has only finitely many nonzero terms, its radius of convergence is $1/(\limsup_{j\to\infty}(0)) = 1/0$, which is interpreted as infinity. That is, the radius of convergence is $R = \infty$. However, this Taylor series agrees with f only when $x \leq 1$. When $x \leq 1$, we have $f(x) = |x-1| = 1-x$. However, when $x > 1$, we have $f(x) = |x-1| = x-1$, and $x - 1 \neq 1 - x$ since $x > 1$. There does not exist any Taylor series that agrees with f on all of $\mathbf{R}$. If such a Taylor series existed, it would have an infinite radius of convergence, and it would be an analytic function, by Proposition 8.13 in the second set of notes. That is, the Taylor series would be infinitely differentiable. However, f is not differentiable at $x = 1$, so f cannot be equal to any such Taylor series.