Analysis 2, Winter 2015, UCLA Instructor: Steven Heilman

Name: ________________ UCLA ID: ____________ Date: ____________

Signature: ________________.
(By signing here, I certify that I have taken this test while refraining from cheating.)

Mid-Term 2

This exam contains 8 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- You have 50 minutes to complete the exam, starting at the beginning of class.
- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Do not write in the table to the right. Good luck!

Reference sheet

Below are some definitions that may be relevant.

Let (X, d) be a metric space. We say that (X, d) is **complete** if and only if the following property holds. For any Cauchy sequence $(x^{(j)})_{j=k}^{\infty}$ of elements of X, then there exists some $x \in X$ such that $(x^{(j)})_{j=k}^{\infty}$ converges to x with respect to d.

Let (X, d_X) and (Y, d_Y) be metric spaces. Let $(f_j)_{j=1}^{\infty}$ be a sequence of functions from X to Y. Let $f : X \to Y$ be another function. We say that $(f_j)_{j=1}^{\infty}$ **converges pointwise** to f on X if and only if, for every $x \in X$, we have

$$\lim_{j\to\infty} f_j(x) = f(x).$$

That is, for all $x \in X$, we have

$$\lim_{j\to\infty} d_Y(f_j(x), f(x)) = 0.$$

That is, for every $x \in X$ and for every $\varepsilon > 0$, there exists $J > 0$ such that, for all $j > J$, we have $d_Y(f_j(x), f(x)) < \varepsilon$.

Let (X, d_X) and (Y, d_Y) be metric spaces. Let $(f_j)_{j=1}^{\infty}$ be a sequence of functions from X to Y. Let $f : X \to Y$ be another function. We say that $(f_j)_{j=1}^{\infty}$ **converges uniformly** to f on X if and only if, for every $\varepsilon > 0$, there exists $J > 0$ such that, for all $j > J$ and for all $x \in X$ we have $d_Y(f_j(x), f(x)) < \varepsilon$.

Let $\sum_{j=0}^{\infty} a_j(x-a)^j$ be a formal power series. The **radius of convergence** $R \geq 0$ of this series is defined to be

$$R := \frac{1}{\limsup_{j\to\infty} |a_j|^{1/j}}.$$

Let E be a subset of $\mathbf{R}$. We say that a function $f : E \to \mathbf{R}$ is **once differentiable on** E if and only if f is differentiable on E. More generally, for any integer $k \geq 2$, we say that $f : E \to \mathbf{R}$ is k **times differentiable on** E, or just k **times differentiable**, if and only if f is differentiable and f' is $k-1$ times differentiable. If f is k times differentiable, we define the k^{th} derivative $f^{(k)} : E \to \mathbf{R}$ by the recursive rule $f^{(1)} := f'$ and $f^{(k)} := (f^{(k-1)})'$, for all $k \geq 2$. We also define $f^{(0)} := f$. A function is said to be **infinitely differentiable** if and only if f is k times differentiable for every $k \geq 0$.

A function $f : \mathbf{R} \to \mathbf{C}$ is $\mathbf{Z}$-periodic if and only if $f(x+k) = f(x)$ for all $x \in \mathbf{R}$ and for all $k \in \mathbf{Z}$. The space of all complex-valued $\mathbf{Z}$-periodic functions is denoted by $C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$.

1. (10 points) Let (X, d_X) be a metric space. For each positive integer j, let $f_j \colon X \to \mathbf{R}$ be a continuous function. (As usual, $\mathbf{R}$ denotes the real line with the standard metric $d(a, b) := |a - b|$, where $a, b \in \mathbf{R}$.) Suppose $(f_j)_{j=1}^{\infty}$ converges pointwise to a function $f \colon X \to \mathbf{R}$. Let $h \colon \mathbf{R} \to \mathbf{R}$ be a continuous function. Show that the sequence of functions $(h \circ f_j)_{j=1}^{\infty}$ converges pointwise to $h \circ f \colon X \to \mathbf{R}$. (As usual, $h \circ f_j(x) := h(f_j(x))$, and $h \circ f(x) := h(f(x))$ for all $j \geq 1$, for all $x \in X$.)

2. (10 points) Let $C([0,1];\mathbf{R})$ denote the set of continuous functions with domain $[0,1]$ and range $\mathbf{R}$. As usual, we consider $C([0,1];\mathbf{R})$ to be a metric space with the metric $d(f,g) := \sup_{x\in[0,1]} |f(x) - g(x)|$, where $f, g \in C([0,1];\mathbf{R})$. Let V denote the subset of $C([0,1];\mathbf{R})$ consisting of all functions $f\colon [0,1] \to \mathbf{R}$ such that $f(0) = 4f(1)$. Is V a complete subset of $C([0,1];\mathbf{R})$? Prove your assertion.

3. (10 points) Find a power series centered at the origin for the function $\tan^{-1}\colon \mathbf{R} \to (-\pi/2/\pi/2)$. Indicate the radius of convergence of this power series and justify your reasoning. (You may assume that $\tan^{-1}$ is differentiable, and that $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$.)

4. (10 points) Give an example of a function $f\colon \mathbf{R} \to \mathbf{R}$ such that f is not the zero function, f is infinitely differentiable, such that $f(0) = 0$ and such that $f^{(k)}(0) = 0$ for all integers $k \geq 1$. Prove that your function f satisfies these properties.

5. (10 points) Define $f\colon \mathbf{R} \to \mathbf{R}$ by $f(x) := |x-1|$ for all $x \in \mathbf{R}$. What is the Taylor series of f at $x = 0$? What is the radius of convergence of this Taylor series? Where does this Taylor series agree with f? Does there exist any Taylor series that agrees with f on all of $\mathbf{R}$? Justify your answers.

(Scratch paper)