131B Midterm 1 Solutions

1. Question 1

Let (X,d) be a metric space. Let $(x_j)_{j=1}^{\infty}$ and let $(y_j)_{j=1}^{\infty}$ be two sequences in (X,d), and let $x,y\in X$. Assume that $(x_j)_{j=1}^{\infty}$ converges to x, and assume that $(y_j)_{j=1}^{\infty}$ converges to y. Prove that $\lim_{j\to\infty} d(x_j,y_j)=d(x,y)$. (Hint: use the triangle inequality many times.)

Solution. From the triangle inequality, $d(x_j,y_j)\leq d(x_j,x)+d(x,y)+d(y,y_j)$. Also from the triangle inequality, $d(x,y)\leq d(x,x_j)+d(x_j,y_j)+d(y_j,y)$. Putting everything together,

$$d(x,y)\leq d(x,x_j)+d(x_j,y_j)+d(y_j,y)\leq 2d(x,x_j)+2d(y,y_j)+d(x,y).$$

Letting $j\to\infty$, we have $d(x,x_j)\to 0$ and $d(y,y_j)\to 0$, by assumption. Therefore, letting $j\to\infty$ in the above inequalities, and applying the Squeeze Theorem, we have

$$d(x,y)\leq \lim_{j\to\infty} d(x_j,y_j)\leq d(x,y).$$

That is, $d(x,y)=\lim_{j\to\infty} d(x_j,y_j)$.

2. Question 2

Let (X,d) be a metric space, and let $(\mathbf{R}^2,d_{\ell_2})$ denote the Euclidean plane with the usual Euclidean metric. Let $f\colon X\to\mathbf{R}^2$ be a function. We then write f in its components as $f=(f_1,f_2)$, so that, for all $x\in X$, we have $f(x)=(f_1(x),f_2(x))$. In particular, $f_1\colon X\to\mathbf{R}$ and $f_2\colon X\to\mathbf{R}$. (As usual, $\mathbf{R}$ denotes the real line with the standard metric $d(a,b):=|a-b|$, where $a,b\in\mathbf{R}$.)

Prove that $f\colon X\to\mathbf{R}^2$ is continuous if and only if both $f_1\colon X\to\mathbf{R}$ and $f_2\colon X\to\mathbf{R}$ are continuous.

Solution. There are a few ways to do this problem. Here is one way. Let $(x_j)_{j=1}^{\infty}$ be a sequence in X that converges to $x\in X$ with respect to the metric d_X. From Theorem 6.3 in the first set of notes, f is continuous if and only if for all such convergent sequences we have $f(x_j)\to f(x)$ as $j\to\infty$. Equivalently, $\|(f_1(x_j),f_2(x_j))-(f_1(x),f_2(x))\|_{\ell_2}\to 0$ as $j\to\infty$. Recall from Homework 1, Exercise 6 that the metrics d_{ℓ_2} and d_{ℓ_∞} are equivalent. That is, we equivalently have $\|(f_1(x_j),f_2(x_j))-(f_1(x),f_2(x))\|_{\ell_\infty}\to 0$ as $j\to\infty$. Equivalently, $\max_{i=1,2}|f_i(x_j)-f_i(x)|\to 0$ as $j\to\infty$. That is, $f_1(x_j)\to f_1(x)$ and $f_2(x_j)\to f_2(x)$ as $j\to\infty$. From Theorem 6.3 in the first set of notes, since $(x_j)_{j=1}^{\infty}$ is an arbitrary convergent sequence in $\mathbf{R}^2$, the previous sentence is equivalent to both f_1 and f_2 being continuous.

3. Question 3

Let n be a positive integer. Let $(\mathbf{R}^n,d_{\ell_2})$ denote the Euclidean space $\mathbf{R}^n$ with the usual Euclidean metric d_{ℓ_2}. Prove that the following set is compact in $\mathbf{R}^n$:

$$\{(x_1,\ldots,x_n)\in\mathbf{R}^n\colon \sum_{i=1}^n |x_i|=1\}.$$

Solution. Define $f(x_1,\ldots,x_n):=\sum_{i=1}^n |x_i|$. The function $f\colon\mathbf{R}^n\to\mathbf{R}$ is evidently continuous, being a finite sum of continuous functions. (As usual, $\mathbf{R}$ denotes the real line with the standard metric d where $d(a,b)=|a-b|$.) From Proposition 3.15(iv) in the first set of notes, the singleton set $\{1\}$ is a closed subset of $\mathbf{R}$. From Theorem 6.5 in the first set of notes, $f^{-1}(\{1\})$ is therefore closed. The set $f^{-1}(\{1\})$ is evidently also bounded, since if $\sum_{i=1}^{n}|x_i| = 1$, then $\max_{j=1,\ldots,n}|x_j| \leq 1$ as well, so

$$(\sum_{i=1}^{n} x_i^2)^{1/2} \leq (\max_{j=1,\ldots,n}|x_j| \cdot \sum_{i=1}^{n}|x_i|)^{1/2} \leq 1.$$

That is, if $(x_1,\ldots,x_n) \in f^{-1}(\{1\})$, then $(x_1,\ldots,x_n) \in B_{\mathbf{R}^n,d_{\ell_2}}(0,1)$. So, $f^{-1}(\{1\})$ is bounded. In conclusion, the set $f^{-1}(\{1\})$ is both closed and bounded. By Theorem 5.9 in the first set of notes, $f^{-1}(\{1\})$ is therefore compact.

4. Question 4

Let X be a set with more than one element. Define a function $d\colon X \times X \to \mathbf{R}$ so that, for all $x, y \in X$, we have $d(x,x) := 0$, and $d(x,y) := 1$ if $x \neq y$. You may assume that d is a metric on X. Prove that (X,d) is disconnected.

Solution. Let $x \in X$ be any point in X. Define $E := \{x\}$ to be the singleton set $\{x\}$, and define $F := X \smallsetminus \{x\}$ to be the complement of E, i.e. the set of all points in X except for x. By assumption on the set X, we know that the set F is nonempty. We claim that both E and F are open. If this is true, then we are done, since then by definition, (X,d) is disconnected. So, we will conclude by showing that both E and F are open. We in fact prove something stronger. Any single point in X is open. Given this fact, Proposition 3.15(vii) from the first set of notes shows that both E and F are open, since e.g. F is the union of single points, so it is the union of open sets, so F is open. We will therefore conclude by showing that any point $x \in X$ is itself an open set. By the definition of the metric d, note that $B(x,1/2)$ is exactly the singleton set $\{x\}$. That is, $B(x,1/2) = \{x\}$. From Proposition 3.15(iii) in the first set of notes, $B(x,1/2)$ is an open set. Therefore, the single point $\{x\}$ is an open set, and we are done.

5. Question 5

Let (X, d_X) be a metric space. Let $\mathbf{R}$ denote the real line with the standard metric. That is, if $a, b \in R$, we consider the metric d on R where $d(a,b) = |a-b|$. For each positive integer j, let $f_j\colon X \to \mathbf{R}$ be a continuous function. Assume that the sequence $(f_j)_{j=1}^{\infty}$ converges uniformly to a function $f\colon X \to \mathbf{R}$. Show that f is also continuous.

Solution. Let $x, y \in X$. Let $\varepsilon > 0$. We need to find $\delta = \delta(\varepsilon) > 0$ such that, if $d_X(x,y) < \delta$, then $|f(x) - f(y)| < \varepsilon$. Since $(f_j)_{j=1}^{\infty}$ converges uniformly to f, we know that there exists $J = J(\varepsilon)$ such that for all $j \geq J$, and for all $x \in X$, we have $|f_j(x) - f(x)| < \varepsilon/4$. So, choose $j = J$. Since f_j is continuous, there exists $\delta > 0$ such that, if $d_X(x,y) < \delta$, then $|f_j(x) - f_j(y)| < \varepsilon/2$. So, from the triangle inequality, we have

$$|f(x) - f(y)| \leq |f(x) - f_j(x)| + |f_j(x) - f_j(y)| + |f_j(y) - f(y)| \leq \varepsilon/4 + \varepsilon/2 + \varepsilon/4 = \varepsilon.$$

That is, we found the required $\delta > 0$. We conclude that f is continuous, as desired.