Analysis 2, Winter 2015, UCLA — Instructor: Steven Heilman

Name: ________________ UCLA ID: ____________ Date: __________

Signature: ________________.
(By signing here, I certify that I have taken this test while refraining from cheating.)

Mid-Term 1

This exam contains 8 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- You have 50 minutes to complete the exam, starting at the beginning of class.
- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Do not write in the table to the right. Good luck!

Reference sheet

Below are some definitions that may be relevant.

A **metric space** (X, d) is a set X together with a function $d\colon X \times X \to [0, \infty)$ which satisfies the following properties. (i) For all $x \in X$, we have $d(x, x) = 0$. (ii) For all $x, y \in X$ with $x \neq y$, we have $d(x, y) > 0$. (Positivity) (iii) For all $x, y \in X$, we have $d(x, y) = d(y, x)$. (Symmetry) (iv) For all $x, y, z \in X$, we have $d(x, z) \leq d(x, y) + d(y, z)$. (Triangle inequality)

Let (X, d) be a metric space. We say that (X, d) is **complete** if and only if the following property holds. For any Cauchy sequence $(x^{(j)})_{j=k}^{\infty}$ of elements of X, then there exists some $x \in X$ such that $(x^{(j)})_{j=k}^{\infty}$ converges to x with respect to d.

A metric space (X, d) is said to be **compact** if and only if every sequence in (X, d) has at least one convergent subsequence. We say that $Y \subseteq X$ is **compact** if and only if the metric space $(Y, d|_{Y \times Y})$ is compact.

Let (X, d) be a metric space. We say that X is **disconnected** if and only if there exist disjoint open sets V, W in X such that $V \cup W = X$. (Equivalently, X is disconnected if and only if X contains a proper non-empty subset which is both open and closed.) We say that X is **connected** if and only if X is not disconnected. We say that $Y \subseteq X$ is **connected** if and only if the metric space $(Y, d|_{Y \times Y})$ is connected. We say that Y is **disconnected** if and only if the metric space $(Y, d|_{Y \times Y})$ is disconnected.

Let (X, d_X) and (Y, d_Y) be metric spaces. Let $(f_j)_{j=1}^{\infty}$ be a sequence of functions from X to Y. Let $f\colon X \to Y$ be another function. We say that $(f_j)_{j=1}^{\infty}$ **converges pointwise** to f on X if and only if, for every $x \in X$, we have

$$\lim_{j \to \infty} f_j(x) = f(x).$$

That is, for all $x \in X$, we have

$$\lim_{j \to \infty} d_Y(f_j(x), f(x)) = 0.$$

That is, for every $x \in X$ and for every $\varepsilon > 0$, there exists $J > 0$ such that, for all $j > J$, we have $d_Y(f_j(x), f(x)) = 0$.

Let (X, d_X) and (Y, d_Y) be metric spaces. Let $(f_j)_{j=1}^{\infty}$ be a sequence of functions from X to Y. Let $f\colon X \to Y$ be another function. We say that $(f_j)_{j=1}^{\infty}$ **converges uniformly** to f on X if and only if, for every $\varepsilon > 0$, there exists $J > 0$ such that, for all $j > J$ and for all $x \in X$ we have $d_Y(f_j(x), f(x)) = 0$.

1. (10 points) Let (X, d) be a metric space. Let $(x_j)_{j=1}^{\infty}$ and let $(y_j)_{j=1}^{\infty}$ be two sequences in (X, d), and let $x, y \in X$. Assume that $(x_j)_{j=1}^{\infty}$ converges to x, and assume that $(y_j)_{j=1}^{\infty}$ converges to y. Prove that $\lim_{j\to\infty} d(x_j, y_j) = d(x, y)$. (Hint: use the triangle inequality many times.)

2. (10 points) Let (X, d) be a metric space, and let $(\mathbf{R}^2, d_{\ell_2})$ denote the Euclidean plane with the usual Euclidean metric. Let $f\colon X \to \mathbf{R}^2$ be a function. We then write f in its components as $f = (f_1, f_2)$, so that, for all $x \in X$, we have $f(x) = (f_1(x), f_2(x))$. In particular, $f_1\colon X \to \mathbf{R}$ and $f_2\colon X \to \mathbf{R}$. (As usual, $\mathbf{R}$ denotes the real line with the standard metric $d(a, b) := |a - b|$, where $a, b \in \mathbf{R}$.)

   Prove that $f\colon X \to \mathbf{R}^2$ is continuous if and only if both $f_1\colon X \to \mathbf{R}$ and $f_2\colon X \to \mathbf{R}$ are continuous.

3. (10 points) Let n be a positive integer. Let $(\mathbf{R}^n, d_{\ell_2})$ denote the Euclidean space $\mathbf{R}^n$ with the usual Euclidean metric d_{ℓ_2}. Prove that the following set is compact in $\mathbf{R}^n$:

$$\{(x_1, \ldots, x_n) \in \mathbf{R}^n \colon \sum_{i=1}^{n} |x_i| = 1\}.$$

4. (10 points) Let X be a set with more than one element. Define a function $d\colon X \times X \to \mathbf{R}$ so that, for all $x, y \in X$, we have $d(x,x) := 0$, and $d(x,y) := 1$ if $x \neq y$. You may assume that d is a metric on X. Prove that (X,d) is disconnected.

5. (10 points) Let (X, d_X) be a metric space. Let $\mathbf{R}$ denote the real line with the standard metric. That is, if $a, b \in R$, we consider the metric d on R where $d(a, b) = |a - b|$. For each positive integer j, let $f_j\colon X \to \mathbf{R}$ be a continuous function. Assume that the sequence $(f_j)_{j=1}^{\infty}$ converges uniformly to a function $f\colon X \to \mathbf{R}$. Show that f is also continuous.

(Scratch paper)