Please provide complete and well-written solutions to the following exercises.

Due October 16, in the discussion section.

Assignment 2

Exercise 1. By breaking into different cases as necessary, prove the following statements. Let x, y be rational numbers. Then $|x| \geq 0$, and $|x| = 0$ if and only if $x = 0$. We also have the **triangle inequality**

$$|x + y| \leq |x| + |y|,$$

the bounds

$$-|x| \leq x \leq |x|$$

and the equality

$$|xy| = |x|\,|y|.$$

In particular,

$$|-x| = |x|.$$

Also, the distance $d(x, y) := |x - y|$ satisfies the following properties. Let x, y, z be rational numbers. Then $d(x, y) = 0$ if and only if $x = y$. Also, $d(x, y) = d(y, x)$. Lastly, we have the triangle inequality

$$d(x, z) \leq d(x, y) + d(y, z).$$

Exercise 2. Using the usual triangle inequality, prove the **reverse triangle inequality**: For any rational numbers x, y, we have $|x - y| \geq ||x| - |y||$.

Exercise 3. Let x be a rational number. Prove that there exists a unique integer n such that $n \leq x < n + 1$. In particular, there exists an integer N such that $x < N$. (Hint: use the Euclidean Algorithm.)

Exercise 4. Let $(a_n)_{n=0}^{\infty}$ be a Cauchy sequence of rationals. Prove that $(a_n)_{n=0}^{\infty}$ is bounded.

Exercise 5. Let $(a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}$ be Cauchy sequences of rationals. Prove that $(a_n b_n)_{n=0}^{\infty}$ is a Cauchy sequence of rationals. In other words, the multiplication of two real numbers gives another real number. Now, let $(a_n')_{n=0}^{\infty}$ be a Cauchy sequence of rationals that is equivalent to $(a_n)_{n=0}^{\infty}$. Prove that $(a_n b_n)_{n=0}^{\infty}$ is equivalent to $(a_n' b_n)_{n=0}^{\infty}$. In other words, multiplication of real numbers is well-defined.

Exercise 6. Let x be a real number and let $\varepsilon > 0$ be any rational number. Show that there exists a rational number y such that $|x - y| < \varepsilon$.

Exercise 7. Let x, z be real numbers with $x < z$. Then there exists a rational number y with $x < y < z$. (Hint: use the previous exercise, and the Archimedean property.)

Exercise 8. Let x be a real number. Show that there exists a Cauchy sequence of rational numbers $(a_n)_{n=0}^{\infty}$ such that $x = \mathrm{LIM}_{n\to\infty} a_n$, and such that $a_n > x$ for all $n \geq 0$.

Exercise 9. For every real number x, show that exactly one of the following statements is true: x is positive, x is negative, or x is zero. Show that if x, y are positive real numbers, then $x + y$ is positive, and xy is positive.

Exercise 10. Let x, y be real numbers. Prove that $(x^2 + y^2)/2 \geq xy$.