131A Final Solutions

1. Question 1

Prove that $\sqrt{2}$ is not a rational number.

Solution. We argue by contradiction. Assume that x is rational and $x^2 = 2$. We may assume that x is positive, since $x^2 = (-x)^2$. Let p, q be integers with $q \neq 0$ such that $x = p/q$. Since x is positive, we may assume that p, q are natural numbers. Since $x^2 = 2$, we have $p^2 = 2q^2$. If $a \in \mathbf{N}$ is odd, then $a = 2b + 1$ for some $b \in \mathbf{N}$, and $a^2 = 4b^2 + 2b + 2b + 1 = 2(2b^2 + b + b) + 1$, so a^2 is odd. So, by taking the contrapositive: if a^2 is even, then a is even. Since $p^2 = 2q^2$, p^2 is even, so we conclude that p is even, so there exists a natural number k such that $p = 2k$. Since p is positive, k is positive. Since $p^2 = 2q^2$, we get $p^2 = 4k^2 = 2q^2$, so $q^2 = 2k^2$. Since $p^2 = 2q^2$, and p, q are positive, we have $q < p$.

In summary, we started with positive natural numbers p, q such that $p^2 = 2q^2$. And we now have positive natural numbers q, k such that $q^2 = 2k^2$, and such that $q < p$. We can therefore iterate this procedure. For any natural number n, suppose inductively we have p_n, q_n positive natural numbers such that $p_n^2 = 2q_n^2$. Then we have found natural numbers p_{n+1}, q_{n+1} such that $p_{n+1}^2 = 2q_{n+1}^2$, and such that $p_{n+1} < p_n$. The existence of the natural numbers $p_1, p_2, \ldots$ violates the principle of infinite descent, so we have obtained a contradiction. We conclude that no rational x satisfies $x^2 = 2$.

2. Question 2

Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers such that $a_{n+1} > a_n$ for all $n \in \mathbf{N}$. Prove that, if m, n are natural numbers such that $m > n$, then $a_m > a_n$.

Solution. We show that $a_{n+k} > a_n$ for all $n \in \mathbf{N}$ and for all $k \in \mathbf{N}$ with $k \geq 1$ by induction on k. The base case is $k = 1$, which says $a_{n+1} > a_n$ for all $n \in \mathbf{N}$. The problem assumes this property, so the base case is verified. Now, assume that $a_{n+k} > a_n$ for all $n \in \mathbf{N}$ for some fixed $k \geq 1$. We will prove that $a_{n+k+1} > a_n$ for all $n \in \mathbf{N}$. By assumption, we have $a_{n+k+1} > a_{n+k}$. Then, by the inductive hypothesis, $a_{n+k} > a_n$. Combining these inequalities, we have $a_{n+k+1} > a_n$ for all $n \in \mathbf{N}$. The inductive step is complete. We conclude that $a_{n+k} > a_n$ for all $n \in \mathbf{N}$ and for all $k \in \mathbf{N}$. Since every natural number $m > n$ is of the form $n + k$ for $k \geq 1$, $k \in \mathbf{N}$, we are done.

3. Question 3

Consider the set $A = \{(x, y) \in \mathbf{R} \times \mathbf{R} \colon x + y \in \mathbf{Q}\}$. Is this set finite, countable, or uncountable? Prove your assertion.

Solution. This set is uncountable. To see this, recall that the real numbers $\mathbf{R}$ are uncountable. Define a function $f \colon \mathbf{R} \to A$ by $f(x) = (x, -x)$ for all $x \in \mathbf{R}$. Note that $f(x)$ is in A for all $x \in \mathbf{R}$, since $x + (-x) = 0 \in \mathbf{Q}$. We now claim that f is a bijection onto its image in A. That is, if we define $f(\mathbf{R}) = \{f(x) \colon x \in \mathbf{R}\} = \{(x, -x) \colon x \in \mathbf{R}\}$, then $f \colon \mathbf{R} \to f(\mathbf{R})$ is a bijection. Indeed, given any element y of $f(\mathbf{R})$ we have $y = (x, -x)$ for some $x \in \mathbf{R}$, so $f(x) = y = (x, -x)$. And this x is unique, since if $f(x) = f(x')$ for some $x, x' \in \mathbf{R}$, then $(x, -x) = (x', -x')$, so that $x = x'$. In conclusion, $f \colon \mathbf{R} \to f(\mathbf{R})$ is a bijection. We now show that A is uncountable. It cannot be the case that A is countable, since A contains the uncountable set $f(\mathbf{R})$. Similarly, A cannot be finite. Therefore, A is uncountable, as desired.

4. Question 4

For the following sequences $(a_n)_{n=1}^{\infty}$, compute $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$. If the limit $\lim_{n\to\infty} a_n$ exists, explain why it exists. If the limit $\lim_{n\to\infty} a_n$ does not exist, explain why it does not exist.

(i) $a_n = (-1)^n$ for all $n \geq 1$, $n \in \mathbf{N}$.

Solution. We have $\limsup_{n\to\infty} a_n = 1$ and $\liminf_{n\to\infty} a_n = -1$. Since the limsup and liminf are not the same, the limit $\lim_{n\to\infty} a_n$ does not exist (by Proposition 6.7(vi) in the second set of notes). To see that these computations are correct, note that $a_n \leq 1$ for all $n \geq 1$, so 1 is always an upper bound for $(a_n)_{n=1}^{\infty}$. However, if $x < 1$, then for any $k \geq 1$, $k \in \mathbf{N}$, we have $a_{2k} = 1 > x$, so x cannot be an upper bound for $(a_n)_{n=2k}^{\infty}$, for any $k \in \mathbf{N}$, $k \geq 1$. We conclude that, for any $n \geq 1$, we have $\sup_{m\geq n} a_m = 1$. Therefore, $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup_{m\geq n} a_m = \lim_{n\to\infty} 1 = 1$. Now, note that $a_n \geq -11$ for all $n \geq 1$, so -1 is always a lower bound for $(a_n)_{n=1}^{\infty}$. However, if $x > -1$, then for any $k \geq 1$, $k \in \mathbf{N}$, we have $a_{2k+1} = -1 < x$, so x cannot be a lower bound for $(a_n)_{n=2k+1}^{\infty}$, for any $k \in \mathbf{N}$, $k \geq 1$. We conclude that, for any $n \geq 1$, we have $\inf_{m\geq n} a_m = -1$. Therefore, $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \inf_{m\geq n} a_m = \lim_{n\to\infty} -1 = -1$.

(ii)$a_n = 1/n$ for all $n \geq 1$, $n \in \mathbf{N}$.

Solution. We know that $\lim_{n\to\infty} 1/n = 0$. So, from Proposition 6.7(vi) in the second set of notes, we have $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = 0$.

(iii)$a_n = n$ for all $n \geq 1$, $n \in \mathbf{N}$.

Solution. For any $N \in \mathbf{N}$, we have $a_n \geq N$ for all $n \geq N$. Therefore, by the definition of supremum and infimum, for any $n \in \mathbf{N}$ we have $\sup_{m\geq n} a_m = +\infty$ and $\inf_{m\geq n} a_m = +\infty$. Therefore, $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = +\infty$. Also, by the definition of the limit, since $a_n \geq N$ for all $n \geq N$, we know that the limit $\lim_{n\to\infty} a_n$ does not exist.

5. Question 5

Determine which of the following series converges. Justify your answer

(i) $\displaystyle\sum_{n=1}^{\infty} (-1)^n$.

Solution. This series does not converge. Note that $(-1)^n$ does not converge to zero as $n \to \infty$. So, by the Zero Test, this series does not converge.

(ii) $\displaystyle\sum_{n=1}^{\infty} 2^{((-1)^n - n)}$.

Solution. This series converges. Note that $(-1)^n - n \leq 1 - n$ for all $n \geq 1$. So, we have $\left|2^{((-1)^n-n)}\right| = 2^{((-1)^n-n)} \leq 2^{1-n}$. Also, we know from geometric series that $\sum_{n=1}^{\infty} 2^{1-n} = 2\sum_{n=1}^{\infty} 2^{-n} = 2 < \infty$. So, by the comparison test, $\sum_{n=1}^{\infty} 2^{((-1)^n-n)}$ converges.

(iii) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{(-1)^n - 3}\right)^n$.

Solution. This series does not converge. Let n be an even natural number. Then $2/((-1)^n - 3) = 2/(-2) = -1$, so $\left(\frac{2}{(-1)^n-3}\right)^n = (-1)^n$. Therefore, if $a_n = 2/((-1)^n - 3)$. Then it does not hold that $\lim_{n\to\infty} a_n = 0$. So, by the Zero Test, the series $\sum_{n=1}^{\infty} a_n$ does not converge.

6. Question 6

(i) Find a continuous function $f\colon \mathbf{R} \to \mathbf{R}$ such that f is differentiable on $\mathbf{R} \setminus \{0\}$, and such that f is not differentiable at 0.

Solution. Consider $f(x) = |x|$ where $x \in \mathbf{R}$. We showed in Example 4.9 in the third set of notes that f is continuous on $\mathbf{R}$ and f is differentiable on $\mathbf{R} \setminus \{0\}$, but f is not differentiable at 0.

(ii) Find a function $f\colon [0, +\infty) \to \mathbf{R}$ which is continuous and bounded, where f attains its maximum somewhere, but f does not attain its minimum anywhere.

Solution. Consider $f(x) = 1/(x+1)$, where $x \geq 0$. Since f is a rational function where the denominator does not vanish on $[0, +\infty)$, we know that f is continuous on $[0, +\infty)$. (See Remark 3.8 in the third set of notes.) Moreover, f is differentiable on $[0, +\infty)$ by the quotient rule. Specifically, $f'(x) = -1/(x+1)^2 < 0$ for all $x \geq 0$. So, f is strictly decreasing on $[0, +\infty)$. (See Proposition 4.32 in the third set of notes.) Therefore, f must attain its maximum at $x = 0$. However, f does not attain its minimum anywhere. Given any $x \in [0, +\infty)$, it cannot be the case that $f(x) \leq f(y)$ for all $y \in [0, +\infty)$, since if $y > x$, then $f(x) > f(y)$ (since f is strictly monotone decreasing).

(iii) Find a function $f\colon [-1, 1] \to \mathbf{R}$ such that $f(-1) \neq f(1)$, such that f is differentiable on $(-1, 1)$, and such that $f'(x) = 0$ for all $x \in (-1, 1)$.

Solution. Define $f\colon [-1, 1] \to \mathbf{R}$ by $f(x) = 0$ for all $x \in [-1, 1)$, and $f(1) = 1$. Then $f(0) = 0 \neq 1 = f(1)$. Since $f|_{(-1,1)}$ is the constant function 0, then f is differentiable on $(-1, 1)$ with $f'(x) = 0$ for all $x \in (-1, 1)$.

7. Question 7

Let q be a positive rational number. Let n be a positive integer. For a real number x, define $f(x) := 1/x^q$. The quantity $\int_{1/n}^1 f$ is increasing in n, so it either has some finite limit as $n \to \infty$, or it diverges. For what values of q does $\int_{1/n}^1 f$ converge to a finite value as $n \to \infty$?

Solution. The integral converges to a finite value as $n \to \infty$ if and only if $0 < q < 1$.

If $q \neq 1$, define $g\colon (0, 1] \to \mathbf{R}$ by $g(x) = (1-q)^{-1} x^{1-q}$. Then $g'(x) = x^{-q} = f(x)$ for all $x \in (0, 1]$. So, by the first fundamental theorem of calculus, we have $\int_{1/n}^1 f = \int_{1/n}^1 g' = g(1) - g(1/n) = (1-q)^{-1}(1 - n^{q-1})$. Letting $n \to \infty$, we see that if $0 < q < 1$, then $\lim_{n\to\infty} \int_{1/n}^1 f = (1-q)^{-1} < \infty$. And if $q > 1$, we have $\lim_{n\to\infty} \int_{1/n}^1 f = (q-1)^{-1}(n^{q-1} - 1) = +\infty$.

The only remaining cases is $q = 1$. In this case, we could use some properties of logarithms, but there is also something simpler to do. Define a function $h\colon (0, 1] \to \mathbf{R}$ so that, for any $n \in \mathbf{N}$, we have $h(x) := 2^n$ whenever x lies in the interval $(2^{-n-1}, 2^{-n}]$. For example, $h(x) = 1$ when $x \in (1/2, 1]$, and $h(x) = 2$ whenever $x \in (1/4, 1/2]$, and $h(x) = 4$ whenever $x \in (1/8, 1/4]$, and so on. In the case $q = 1$, we have $f(x) = 1/x$, so that $f'(x) = -1/x^2$, and f is strictly decreasing on $(0, 1]$. So, for any $n \in \mathbf{N}$, if $x \in (2^{-n-1}, 2^{-n}]$, we have $f(x) \geq f(2^{-n}) = 2^n$. That is, if $x \in (2^{-n-1}, 2^{-n}]$, we have $f(x) \geq h(x)$. From the monotonic property of integrals, we therefore have

$$\int_{2^{-n-1}}^1 f \geq \int_{2^{-n-1}}^1 h = \sum_{k=1}^n (2^{-k} - 2^{-k-1}) 2^k = \sum_{k=1}^n 2^{-k-1} 2^k = \sum_{k=1}^n (1/2) = n/2.$$

Therefore, letting $n \to \infty$, we have

$$\lim_{n\to\infty} \int_{1/n}^{1} f = \lim_{n\to\infty} \int_{2^{-n}}^{1} f \geq \lim_{n\to\infty} (n/2) = +\infty.$$

8. Question 8

Let $a < b$ be real numbers. Let $f\colon [a,b] \to \mathbf{R}$ be a Riemann integrable function such that $f(x) = 0$ whenever x is a rational number. Prove that $\int_a^b f = 0$.

Solution. Let $a = x_0 < x_1 < \cdots < x_n = b$ be a partition of $[a,b]$. For any $i \in \{1,\ldots,n\}$, note that there always exists a rational number in the interval $[x_{i-1}, x_i]$, by the density of rationals in $\mathbf{R}$. Therefore, for any $i \in \{1,\ldots,n\}$, we have

$$\inf_{x\in[x_{i-1},x_i]} f(x) \leq 0.$$

$$\sup_{x\in[x_{i-1},x_i]} f(x) \geq 0.$$

We therefore estimate

$$L(f,P) = \sum_{i=1}^{n} (\inf_{x\in[x_{i-1},x_i]} f(x))(x_i - x_{i-1}) \leq \sum_{i=1}^{n} 0 \cdot (x_i - x_{i-1}) \leq 0.$$

$$U(f,P) = \sum_{i=1}^{n} (\sup_{x\in[x_{i-1},x_i]} f(x))(x_i - x_{i-1}) \geq \sum_{i=1}^{n} 0 \cdot (x_i - x_{i-1}) \geq 0.$$

So, using the definition of lower and upper Riemann integrals, we have

$$\underline{\int_a^b} f \leq 0.$$

$$\overline{\int_a^b} f \geq 0.$$

Since f is Riemann integrable, by definition we must have $\underline{\int_a^b} f = \overline{\int_a^b} f = \int_a^b f$. That is, $0 \leq \int_a^b f \leq 0$. In conslusion, $\int_a^b f = 0$.

9. Question 9

Let x, t be a real numbers. Let

$$g(x,t) := \frac{1}{1 + xt^4}.$$

Suppose we integrate g with respect to x, and we define

$$f(t) := \int_0^1 g = \int_0^1 \frac{1}{1 + xt^4} dx.$$

Prove that f is continuous at $t = 0$.

Solution 1. Note that $f(0) = \int_0^1 1 = 1$. So, we need to show that, as $t \to 0$, we have $f(t) \to 1$. Let $\varepsilon > 0$. We need to find $\delta > 0$ such that $|f(t) - 1| < \varepsilon$ whenever $|t| < \delta$. Let $|t| < \delta$. For fixed t, consider the function $G(x) = g(x,t)$ where $x \in [0,1]$. Since $|t| < 1/2$, we have $|t|^4 < 1/16$, so $|xt^4| < |x|/16 < 1/16 < 1$ whenever $x \in [0,1]$. So, the quantity $1 + xt^4$

does not vanish for $x \in [0,1]$, so G is continuous, differentiable, and Riemann integrable for all $x \in [0,1]$. Note that $G'(x) = -t^4/(1+xt^4)^2$, so G is strictly decreasing on $[0,1]$. That is, for all $x \in [0,1]$, we have

$$G(1) \leq G(x) \leq G(0), \quad \forall\, x \in [-1,1]$$

That is,

$$\frac{1}{1+t^4} \leq G(x) \leq 1, \quad \forall\, x \in [-1,1].$$

Since $|t| < \delta < 1/2$, and the function $1/(1+t^4)$ is similarly monotone in t, we have

$$\frac{1}{1+\delta^4} \leq \frac{1}{1+t^4}.$$

In conclusion, if $|t| < \delta$ and if $x \in [0,1]$, we have

$$\frac{1}{1+\delta^4} \leq G(x) \leq 1.$$

Since $\lim_{\delta \to 0} 1/(1+\delta^4) = 1$, there exists $\gamma > 0$ such that, if $|\delta| < \gamma$, we have $1-\varepsilon < 1/(1+\delta^4)$. In summary, if $|t| < \gamma$, then for all $x \in [-1,1]$, we have

$$1 - \varepsilon \leq G(x) \leq 1.$$

Integrating this inequality, over $[-1,1]$, we have

$$1 - \varepsilon \leq \int_0^1 G(x) \leq 1.$$

That is, if $|t| < \gamma$, then $|f(t) - 1| < \varepsilon$. So, f is continuous at $t = 0$, as desired.

Solution 2. (Assuming use of change of variables.) Changing variables $y = t^4 x$, we get

$$f(t) = t^{-4} \int_0^{t^4} \frac{1}{1+y} dy$$

Note that $H(t) := \int_0^t (1/(1+y))dy$ is a continuous function of t by the second part of the fundamental theorem of calculus. So, $H(t^4)$ is a continuous function of t, since it is a composition of continuous functions. Then $f(t) = t^{-4}H(t^4)$ is a product of functions which are continuous at $t = 1$, so f is continuous at $t = 1$.

10. Question 10

A subset X of the real numbers is said to have measure zero if and only if the following condition is satisfied: Given any $\varepsilon > 0$, there exists a countable set of open intervals $(a_0, b_0), (a_1, b_1), \ldots$ with $a_i < b_i$ for all $i \in \mathbf{N}$, such that $X \subseteq \bigcup_{i=0}^{\infty} (a_i, b_i)$, and such that $\sum_{i=0}^{\infty} (b_i - a_i) < \varepsilon$.

Show that the rational numbers $\mathbf{Q}$ have measure zero.

Solution. Recall that the rational numbers $\mathbf{Q}$ are countable. so, let $f\colon \mathbf{N} \to \mathbf{Q}$ be a bijection. Let $\varepsilon > 0$. For any $i \in \mathbf{N}$, consider the open interval $(f(i) - \varepsilon 2^{-i}, f(i) + \varepsilon 2^{-i})$. That is, define $a_i = f(i) - \varepsilon 2^{-i}$ and $b_i = f(i) + \varepsilon 2^{-i}$. Then $b_i > a_i$ and $b_i - a_i = 2\varepsilon 2^{-i}$. Since the rational number $f(i)$ is contained in the interval $(f(i) - \varepsilon 2^{-i}, f(i) + \varepsilon 2^{-i})$ for each $i \in \mathbf{N}$, and since $\mathbf{Q}$ is equal to the set $\{f(i) \colon i \in \mathbf{N}\}$ (since f is a bijection), we conclude that

$$\mathbf{Q} \subseteq \bigcup_{i=0}^{\infty} (a_i, b_i).$$

Lastly, note that

$$\sum_{i=0}^{\infty} (b_i - a_i) = \sum_{i=0}^{\infty} 2\varepsilon 2^{-i} = 2\varepsilon \sum_{i=0}^{\infty} 2^{-i} = 4\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we are done.