Analysis 1, Fall 2014, UCLA Instructor: Steven Heilman

Name: ____________________ UCLA ID: ______________ Date: ____________

Signature: ____________________.
(By signing here, I certify that I have taken this test while refraining from cheating.)

Final Exam

This exam contains 16 pages (including this cover page) and 10 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- You have 180 minutes to complete the exam.
- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Do not write in the table to the right. Good luck!

Problem	Points	Score
1	10	
2	10	
3	10	
4	15	
5	15	
6	15	
7	15	
8	15	
9	15	
10	10	
Total:	130	

Reference sheet

Below are some definitions that may be relevant. The topics are ordered chronologically. Recall that $\mathbf{N}$ denotes the set of natural numbers, $\mathbf{Q}$ denotes the set of rational numbers, and $\mathbf{R}$ denotes the set of real numbers.

Let X, Y be sets. A **bijection** is a function $f\colon X \to Y$ such that, for all $y \in Y$, there exists exactly one $x \in X$ such that $f(x) = y$. We say that X is **finite** if and only if there exists $n \in \mathbf{N}$ such that there exists a bijection $f\colon X \to \{1 \leq i \leq n \colon i \in \mathbf{N}\}$. We say that X is **countable** if and only if there exists a bijection $f\colon X \to \mathbf{N}$. We say that X is **uncountable** if and only if X is not finite and X is not countable.

Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers, and let L be a real number. We say that the sequence $(a_n)_{n=0}^{\infty}$ **converges to** L if and only if, for every real $\varepsilon > 0$, there exists a natural number $N = N(\varepsilon)$ such that, for all $n \geq N$, we have $|a_n - L| < \varepsilon$.

Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers that is converging to a real number L. We then say that the sequence $(a_n)_{n=0}^{\infty}$ is **convergent**, and we write $L = \lim_{n\to\infty} a_n$. If $(a_n)_{n=0}^{\infty}$ is not convergent, we say that the sequence $(a_n)_{n=0}^{\infty}$ is **divergent**, and we say the limit of L is undefined.

Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers. We say that $(a_n)_{n=0}^{\infty}$ is a **Cauchy sequence** if and only if, for any real $\varepsilon > 0$, there exists a natural number $N = N(\varepsilon)$ such that, for all $n, m \geq N$, we have $|a_n - a_m| < \varepsilon$.

A sequence $(a_n)_{n=0}^{\infty}$ of real numbers is **bounded** if and only if there exists $M \in \mathbf{R}$ such that $|a_n| \leq M$ for all $n \in \mathbf{N}$.

Let E be a subset of $\mathbf{R}$ with some upper bound. The least upper bound of E is called the **supremum** of E, and is denoted by $\sup(E)$ or $\sup E$. If E has no upper bound, we write $\sup(E) = +\infty$. If E is empty, we write $\sup(E) = -\infty$. Let E be a subset of $\mathbf{R}$ with some lower bound. The greatest lower bound of E is called the **infimum** of E, and is denoted by $\inf(E)$ or $\inf E$. If E has no lower bound, we write $\inf(E) = -\infty$. If E is empty, we write $\inf(E) = +\infty$.

Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers. Define $\sup(a_n)_{n=m}^{\infty}$ to be the supremum of the set $\{a_n \colon n \geq m,\, n \in \mathbf{N}\}$. Define $\inf(a_n)_{n=m}^{\infty}$ to be the infimum of the set $\{a_n \colon n \geq m,\, n \in \mathbf{N}\}$.

Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers and let x be a real number. We say that x is a **limit point** of the sequence $(a_n)_{n=m}^{\infty}$ if and only if: for every real $\varepsilon > 0$, for every natural number $N \geq m$, there exists $n \geq N$ such that $|a_n - x| < \varepsilon$. We define

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup_{m\geq n} a_m = \inf_{n\geq m} \sup_{t\geq n} a_t.$$

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf_{m\geq n} a_m = \sup_{n\geq m} \inf_{t\geq n} a_t.$$

Let $\sum_{n=m}^{\infty} a_n$ be a formal infinite series. For any integer $N \geq m$, define the N^{th} **partial sum** S_N of this series by $S_N := \sum_{n=m}^{N} a_n$. If the sequence $(S_N)_{N=m}^{\infty}$ converges to some limit $L \in \mathbf{R}$ as $N \to \infty$, then we say that the infinite series $\sum_{n=m}^{\infty} a_n$ is **convergent**, and this infinite series **converges to** L. We say that the series $\sum_{n=m}^{\infty} a_n$ is **absolutely convergent** if and only if the series $\sum_{n=m}^{\infty} |a_n|$ is convergent. If a series is not absolutely convergent, then it is absolutely divergent.

Zero Test. Let $\sum_{n=m}^{\infty} a_n$ be a formal series of real numbers. If $\sum_{n=m}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. Note that the contrapositive says: if a_n does not converge to zero as $n \to \infty$, then $\sum_{n=m}^{\infty} a_n$ does not converge.

Alternating Series Test. Let $(a_n)_{n=m}^{\infty}$ be a decreasing sequence of nonnegative real numbers. That is, $a_{n+1} \leq a_n$ and $a_n \geq 0$ for all $n \geq m$. Then the series $\sum_{n=m}^{\infty} (-1)^n a_n$ converges if and only if $a_n \to 0$ as $n \to \infty$.

Comparison Test. Let $\sum_{n=m}^{\infty} a_n, \sum_{n=m}^{\infty} b_n$ be formal series of real numbers. Assume that $|a_n| \leq b_n$ for all $n \geq m$. If $\sum_{n=m}^{\infty} b_n$ is convergent, then $\sum_{n=m}^{\infty} a_n$ is absolutely convergent. Moreover, $|\sum_{n=m}^{\infty} a_n| \leq \sum_{n=m}^{\infty} |a_n| \leq \sum_{n=m}^{\infty} b_n$.

The Root Test. Let $\sum_{n=m}^{\infty} a_n$ be a series of real numbers. Define $\alpha := \limsup_{n\to\infty} |a_n|^{1/n}$. (i) If $\alpha < 1$, then the series $\sum_{n=m}^{\infty} a_n$ is absolutely convergent. In particular, the series $\sum_{n=m}^{\infty} a_n$ is convergent. (ii) If $\alpha > 1$, then the series $\sum_{n=m}^{\infty} a_n$ is divergent. (iii) If $\alpha = 1$, no conclusion is asserted.

The Ratio Test. Let $\sum_{n=m}^{\infty} a_n$ be a series of nonzero numbers. (So, a_{n+1}/a_n is defined for any $n \geq m$.) (i) If $\limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$, then the series $\sum_{n=m}^{\infty} a_n$ is absolutely convergent. In particular, $\sum_{n=m}^{\infty} a_n$ is convergent. (ii) If $\liminf_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} > 1$, then the series $\sum_{n=m}^{\infty} a_n$ is divergent. In particular, $\sum_{n=m}^{\infty} a_n$ is not absolutely convergent.

Let X be a subset of $\mathbf{R}$ and let $f\colon X \to \mathbf{R}$ be a function. Let x_0 be an element of X. We say that f is **continuous** at x_0 if and only if $\lim_{x\to x_0; x\in X} f(x) = f(x_0)$. That is, the limit of f at x_0 in X exists, and this limit is equal to $f(x_0)$. We say that f is **continuous on** X (or we just say that f is **continuous**) if and only if f is continuous at x_0 for every $x_0 \in X$. We say that f is **uniformly continuous** if and only if, for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $x \in X$ satisfies $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$. We say that f is **Lipschitz continuous** with constant L if and only if there exists $L \geq 0$ such that, for every $x, y \in X$, we have $|f(x) - f(y)| \leq L\,|x - y|$.

Let $f\colon X \to \mathbf{R}$ be a function, and let $x_0 \in X$. We say that f **attains its maximum** at x_0 if and only if $f(x_0) \geq f(x)$ for all $x \in X$. We say that f **attains its minimum** at x_0 if and only if $f(x_0) \leq f(x)$ for all $x \in X$.

The Maximum Principle. Let $a < b$ be real numbers and let $f\colon [a, b] \to \mathbf{R}$ be a function that is continuous on $[a, b]$. Then f attains its maximum and minimum on $[a, b]$.

Intermediate Value Theorem. Let $a < b$ be real numbers. Let $f\colon [a, b] \to \mathbf{R}$ be function that is continuous on $[a, b]$. Let y be a real number between $f(a)$ and $f(b)$, so that either $f(a) \leq y \leq f(b)$ or $f(a) \geq y \geq f(b)$. Then there exists a $c \in [a, b]$ such that $f(c) = y$.

Let X be a subset of $\mathbf{R}$ and let x be a real number. We say that x is a **limit point** of X if and only if, for every real $\varepsilon > 0$, there exists a $y \in X$ with $y \neq x$ such that $|y - x| < \varepsilon$.

Let X be a subset of $\mathbf{R}$, and let x_0 be an element of X which is also a limit point of X. Let $f \colon X \to \mathbf{R}$ be a function. If the limit $\lim_{x \to x_0; x \in X \setminus \{x_0\}} \frac{f(x) - f(x_0)}{x - x_0}$ converges to a real number L, then we say that f is **differentiable** at x_0 on X **with derivative** L, and we write $f'(x_0) := L$. If this limit does not exist, or if x_0 is not a limit point of X, we leave $f'(x_0)$ undefined, and we say that f is **not differentiable** at x_0 on X.

Mean Value Theorem. Let $a < b$ be real numbers, and let $f \colon [a, b] \to \mathbf{R}$ be a continuous function which is differentiable on (a, b). Then there exists $x \in (a, b)$ such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

Let $a < b$ be real numbers, let $f \colon [a, b] \to \mathbf{R}$ be a bounded function, and let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$. That is, $a = x_0 < x_1 < \cdots < x_n = b$.We define the **upper Riemann sum** $U(f, P)$ by $U(f, P) := \sum_{i=1}^{n} (\sup_{x \in [x_{i-1}, x_i]} f(x))(x_i - x_{i-1})$. We also define the **lower Riemann sum** $L(f, P)$ by $L(f, P) := \sum_{i=1}^{n} (\inf_{x \in [x_{i-1}, x_i]} f(x))(x_i - x_{i-1})$.

Let $a < b$ be real numbers, let $f \colon [a, b] \to \mathbf{R}$ be a bounded function. We define the **upper Riemann integral** $\overline{\int_a^b} f$ of f on $[a, b]$ by

$$\overline{\int_a^b} f := \inf\{U(f, P) \colon P \text{ is a partition of } [a, b]\}.$$

We also define the **lower Riemann integral** $\underline{\int_a^b} f$ of f on $[a, b]$ by

$$\underline{\int_a^b} f := \sup\{L(f, P) \colon P \text{ is a partition of } [a, b]\}.$$

Let $a < b$ be real numbers, let $f \colon [a, b] \to \mathbf{R}$ be a bounded function. If $\overline{\int_a^b} f = \underline{\int_a^b} f$ we say that f is **Riemann integrable** on $[a, b]$, and we define $\int_a^b f := \overline{\int_a^b} f = \underline{\int_a^b} f$.

Fundamental Theorem of Calculus, Part 1. Let $a < b$ be real numbers. Let $f \colon [a, b] \to \mathbf{R}$ be a continuous function on $[a, b]$. Assume that f is also differentiable on $[a, b]$, and f' is Riemann integrable on $[a, b]$. Then $\int_a^b f' = f(b) - f(a)$..

Fundamental Theorem of Calculus, Part 2. Let $a < b$ be real numbers. Let $f \colon [a, b] \to \mathbf{R}$ be a Riemann integrable function. Define a function $F \colon [a, b] \to \mathbf{R}$ by $F(x) := \int_a^x f$. Then F is continuous. Moreover, if $x_0 \in [a, b]$ and if f is continuous at x_0, then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

1. (10 points) Prove that $\sqrt{2}$ is not a rational number.

2. (10 points) Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers such that $a_{n+1} > a_n$ for all $n \in \mathbf{N}$. Prove that, if m, n are natural numbers such that $m > n$, then $a_m > a_n$.

3. (10 points) Consider the set $A = \{(x, y) \in \mathbf{R} \times \mathbf{R} \colon x + y \in \mathbf{Q}\}$. Is this set finite, countable, or uncountable? Prove your assertion.

4. For the following sequences $(a_n)_{n=1}^{\infty}$, compute $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$. If the limit $\lim_{n\to\infty} a_n$ exists, explain why it exists. If the limit $\lim_{n\to\infty} a_n$ does not exist, explain why it does not exist.

   (a) (5 points) $a_n = (-1)^n$ for all $n \geq 1$, $n \in \mathbf{N}$.

   (b) (5 points) $a_n = 1/n$ for all $n \geq 1$, $n \in \mathbf{N}$.

   (c) (5 points) $a_n = n$ for all $n \geq 1$, $n \in \mathbf{N}$.

5. Determine which of the following series converges. Justify your answer

(a) (5 points) $\sum_{n=1}^{\infty} (-1)^n$.

(b) (5 points) $\sum_{n=1}^{\infty} 2^{((-1)^n - n)}$.

(c) (5 points) $\sum_{n=1}^{\infty} \left(\frac{2}{(-1)^n - 3} \right)^n$.

6. (a) (5 points) Find a continuous function $f\colon \mathbf{R} \to \mathbf{R}$ such that f is differentiable on $\mathbf{R} \setminus \{0\}$, and such that f is not differentiable at 0.

(b) (5 points) Find a function $f\colon [0, \infty) \to \mathbf{R}$ which is continuous and bounded, where f attains its maximum somewhere, but f does not attain its minimum anywhere.

(c) (5 points) Find a function $f\colon [-1, 1] \to \mathbf{R}$ such that $f(-1) \neq f(1)$, such that f is differentiable on $(-1, 1)$, and such that $f'(x) = 0$ for all $x \in (-1, 1)$.

7. (15 points) Let q be a positive rational number. Let n be a positive integer. For a real number x, define $f(x) := 1/x^q$. The quantity $\int_{1/n}^{1} f$ is increasing in n, so it either has some finite limit as $n \to \infty$, or it diverges. For what values of q does $\int_{1/n}^{1} f$ converge to a finite value as $n \to \infty$?

8. (15 points) Let $a < b$ be real numbers. Let $f\colon [a,b] \to \mathbf{R}$ be a Riemann integrable function such that $f(x) = 0$ whenever x is a rational number. Prove that $\int_a^b f = 0$.

9. (15 points) Let x, t be a real numbers. Let

$$g(x,t) := \frac{1}{1+xt^4}.$$

Suppose we integrate g with respect to x, and we define

$$f(t) := \int_0^1 g = \int_0^1 \frac{1}{1+xt^4} dx.$$

Prove that f is continuous at $t = 0$.

10. (10 points) A subset X of the real numbers is said to have measure zero if and only if the following condition is satisfied: Given any $\varepsilon > 0$, there exists a countable set of open intervals $(a_0, b_0), (a_1, b_1), \ldots$ with $a_i < b_i$ for all $i \in \mathbf{N}$, such that $X \subseteq \bigcup_{i=0}^{\infty}(a_i, b_i)$, and such that $\sum_{i=0}^{\infty}(b_i - a_i) < \varepsilon$.

    Show that the rational numbers $\mathbf{Q}$ have measure zero. (Hint: as a warmup, try to show that any finite set of rational numbers has measure zero.)

(Scratch paper)

(More scratch paper)