131A Midterm 2 Solutions

1. Question 1

True/False

(a) Let $(a_n)_{n=0}^{\infty}$ be a convergent sequence of real numbers. Then $(a_n)_{n=0}^{\infty}$ is a Cauchy sequence.

TRUE. This was a Theorem 3.10 in the second set of notes.

(b) Let $(a_n)_{n=0}^{\infty}$ be a bounded sequence of real numbers. Then $(a_n)_{n=0}^{\infty}$ is a convergent sequence.

FALSE. The sequence $((-1)^n)_{n=0}^{\infty}$ is bounded by 1, but it is not convergent. To see that this sequence is not convergent, note that it is not Cauchy, since $|(-1)^n - (-1)^{n+1}| = 2$ for all $n \geq 0$. Alternately, note that $\limsup_{n\to\infty}(-1)^n = 1$ while $\liminf_{n\to\infty}(-1)^n = -1$. Since the limsup of the sequence is not equal to the liminf, the sequence is not convergent. (See Proposition 6.7(vi) in the second set of notes.)

(c) Let $(a_n)_{n=0}^{\infty}$ be a positive, decreasing sequence of real numbers. (That is, $a_n \geq 0$ and $a_{n+1} \leq a_n$ for all $n \in \mathbf{N}$.) Then $\sum_{n=0}^{\infty}(-1)^n a_n$ converges.

FALSE. (This is almost the alternating series test, but not quite.) Consider the sequence $a_n = 1$ for all $n \geq 0$. This sequence is positive and decreasing. However, $\sum_{n=0}^{\infty}(-1)^n a_n$ does not converge. To see this, let $S_N = \sum_{n=0}^{N}(-1)^n$ denote the N^{th} partial sum. Then $|S_N - S_{N+1}| = 2$ for all $N \geq 0$. So, the partial sums do not converge, i.e. the sequence $(S_N)_{N=0}^{\infty}$ does not converge as $N \to \infty$. Therefore, the sum $\sum_{n=0}^{\infty}(-1)^n a_n$ does not converge.

(d) Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers such that $\left|\dfrac{a_{n+1}}{a_n}\right| < 1$ for all natural numbers n. Then $\sum_{n=0}^{\infty} a_n$ converges.

FALSE. Let $a_n = (1 + 1/n)$ for all $n \geq 1$. Then for all $n \geq 1$, $|a_{n+1}/a_n| = \frac{1+\frac{1}{n+1}}{1+\frac{1}{n}} < 1$. However, $a_n \to 1$ as $n \to \infty$. So, by the zero test, the series $\sum_{n=1}^{\infty} a_n$ diverges.

2. Question 2

Determine which of the following series converges. Justify your answer

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$.

Let $a_n = n/2^n$ for any $n \geq 1$. We compute: $\limsup_{n\to\infty}|a_{n+1}/a_n| = \limsup_{n\to\infty}\left|\frac{n+1}{2^{n+1}}\frac{2^n}{n}\right| = \limsup_{n\to\infty}\frac{n+1}{n}\frac{1}{2} = \frac{1}{2} < 1$. So, from the ratio test, the series converges.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

Let $a_n = 1/\sqrt{n}$ for all $n \geq 1$. Note that $a_n \geq 0$ for all $n \geq 1$. Also, $a_{n+1} \leq a_n$ since $a_{n+1} = 1/\sqrt{n+1} \leq 1/\sqrt{n} = a_n$. Also, $\lim_{n\to\infty} a_n = 0$. So, from the alternating series test, the series converges.

3. Question 3

Let $f\colon [0,1] \to [0,1]$ be a continuous function. Show that there exists some $x \in [0,1]$ such that $f(x) = x$.

Solution. Consider $g(x) := f(x) - x$ where $g\colon [0,1] \to \mathbf{R}$. Since $f(x) \geq 0$ for all $x \in [0,1]$, we have $g(0) = f(0) \geq 0$. Also, since $f(x) \leq 1$ for all $x \in [0,1]$, we have $g(1) = f(1) - 1 \leq 0$. Note also that since f is continuous, then g is the sum of two continuous functions, so g is continuous. That is, $g\colon [0,1] \to \mathbf{R}$ is continuous with $g(0) \geq 0$ and $g(1) \leq 0$. So, by the Intermediate Value Theorem, there exists some $x \in [0,1]$ such that $g(x) = 0$. By the definition of g, we therefore have $f(x) - x = 0$, i.e. $f(x) = x$, as desired.

4. Question 4

Let $x > 1$. Prove that $\lim_{n\to\infty} \frac{x^n}{n} = +\infty$.

Solution. Write $x = 1 + \varepsilon$ where $\varepsilon > 0$. Let $n \geq 1$. From the binomial theorem,

$$(1+\varepsilon)^n = 1 + n\varepsilon + \varepsilon^2 n(n-1)/2 + r,$$

where $r \geq 0$. That is,

$$(1+\varepsilon)^n \geq 1 + n\varepsilon + \varepsilon^2 n(n-1)/2.$$

Therefore,

$$\frac{x^n}{n} \geq \frac{1 + n\varepsilon + \varepsilon^2 n(n-1)/2}{n} = \frac{1}{n} + \varepsilon + \varepsilon^2(n-1)/2.$$

Taking the liminf of both sides,

$$\liminf_{n\to\infty} \frac{x^n}{n} \geq \liminf_{n\to\infty} \left(\frac{1}{n} + \varepsilon + \varepsilon^2(n-1)/2\right) = +\infty.$$

(Note that we cannot take the limit of both sides, since we do not yet know the limit exists.) Since $\liminf_{n\to\infty} x^n/n = +\infty$, we conclude that $\lim_{n\to\infty} x^n/n = +\infty$.

5. Question 5

Let $x \in \mathbf{R}$. Consider the function $f(x) := \sum_{n=1}^{\infty} \frac{1}{x^2+n^2}$. Prove that f is continuous on $(-\infty, +\infty)$.

Solution. First, note that the sum converges absolutely for any $x \in \mathbf{R}$ by the comparison test, since $1/(x^2+n^2) \leq 1/n^2$ for all $x \in \mathbf{R}$ and for all $n \geq 1$, and we know that $\sum_{n=1}^{\infty} 1/n^2 < \infty$. In particular, for any $x \in \mathbf{R}$, and for any $N \in \mathbf{N}$ we have by the comparison test that

$$\sum_{n=N}^{\infty} \frac{1}{x^2+n^2} \leq \sum_{n=N}^{\infty} \frac{1}{n^2}. \qquad (*)$$

Let $\varepsilon > 0$. We will in fact show that f is uniformly continuous on $\mathbf{R}$, so that in particular f is continuous. We will find $\delta = \delta(\varepsilon) > 0$ such that, whenever $x, y \in \mathbf{R}$ satisfy $|x - y| < \delta$, we have $|f(x) - f(y)| < \varepsilon$.

For each $n \geq 1$, note that the function $f_n(x) := 1/(x^2+n^2)$ is Lipschitz continuous on $\mathbf{R}$. To see this, let $x, y \in \mathbf{R}$ and observe

$$\begin{aligned} |f_n(x) - f_n(y)| &= \left|\frac{1}{x^2+n^2} - \frac{1}{y^2+n^2}\right| = \left|\frac{y^2-x^2}{(x^2+n^2)(y^2+n^2)}\right| \\ &= |x-y| \frac{|x+y|}{(x^2+n^2)(y^2+n^2)} \leq |x-y| \frac{|x+y|}{x^2+y^2+1} \leq |x-y| \frac{2\max(|x|,|y|)}{x^2+y^2+1}. \end{aligned}$$

If $\max(|x|,|y|) \leq 1$, then $x^2+y^2+1 \geq 1$, so $\frac{2\max(|x|,|y|)}{x^2+y^2+1} \leq 2$. If $\max(|x|,|y|) \geq 1$, then $\max(|x|,|y|) \leq \max(|x|^2,|y|^2)$, so $\frac{2\max(|x|,|y|)}{x^2+y^2+1} \leq \frac{2\max(|x|,|y|)}{\max(|x|^2,|y|^2)+1} \leq 2$. In any case, we have

$$|f_n(x) - f_n(y)| \leq 2|x-y|, \forall x, y \in \mathbf{R}, \quad \forall n \in \mathbf{N} \qquad (**)$$

That is, f_n is Lipschitz continuous with constant 2 for every $n \geq 1$.

Now, let $x, y \in \mathbf{R}$. Then, for any $N \in \mathbf{N}$, by $(**)$ we have

$$\left|\sum_{n=1}^{N} f_n(x) - \sum_{n=1}^{N} f_n(y)\right| \leq \sum_{n=1}^{N} |f_n(x) - f_n(y)| \leq 2N|x-y| \qquad (***).$$

So, first choose N so that $\sum_{n=N}^{\infty} \frac{1}{n^2} < \varepsilon/4$ (which is possible since this series converges). Then, choose $\delta := \varepsilon/(4N)$, and let $|x-y| < \delta$. Combining $(*)$ and $(***)$, we get

$$\begin{aligned}
|f(x) - f(y)| &= \left|\sum_{n=1}^{N}(f_n(x) - f_n(y)) + \sum_{n=N+1}^{\infty}(f_n(x) - f_n(y))\right| \\
&\leq \left|\sum_{n=1}^{N}(f_n(x) - f_n(y))\right| + \left|\sum_{n=N+1}^{\infty} f_n(x)\right| + \left|\sum_{n=N+1}^{\infty} f_n(y)\right| \\
&\leq 2N|x-y| + \varepsilon/4 + \varepsilon/4 \leq 2N\delta + \varepsilon/2 \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

In conclusion, f is uniformly continuous, so f is continuous.