Analysis 1, Fall 2014, UCLA — Instructor: Steven Heilman

Name: ____________________ UCLA ID: ______________ Date: ____________

Signature: ____________________.
(By signing here, I certify that I have taken this test while refraining from cheating.)

Mid-Term 2

This exam contains 8 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- You have 50 minutes to complete the exam, starting at the beginning of class.
- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Do not write in the table to the right. Good luck!

Reference sheet

Below are some definitions that may be relevant.

Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers, and let L be a real number. We say that the sequence $(a_n)_{n=0}^{\infty}$ **converges to** L if and only if, for every real $\varepsilon > 0$, there exists a natural number $N = N(\varepsilon)$ such that, for all $n \geq N$, we have $|a_n - L| < \varepsilon$.

Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers. We say that $(a_n)_{n=0}^{\infty}$ is a **Cauchy sequence** if and only if, for any real $\varepsilon > 0$, there exists a natural number $N = N(\varepsilon)$ such that, for all $n, m \geq N$, we have $|a_n - a_m| < \varepsilon$.

A sequence $(a_n)_{n=0}^{\infty}$ of real numbers is **bounded** if and only if there exists $M \in \mathbf{R}$ such that $|a_n| \leq M$ for all $n \in \mathbf{N}$.

Let $\sum_{n=m}^{\infty} a_n$ be a formal infinite series. For any integer $N \geq m$, define the N^{th} **partial sum** S_N of this series by $S_N := \sum_{n=m}^{N} a_n$. If the sequence $(S_N)_{N=m}^{\infty}$ converges to some limit $L \in \mathbf{R}$ as $N \to \infty$, then we say that the infinite series $\sum_{n=m}^{\infty} a_n$ is **convergent**, and this infinite series **converges to** L.

The Root Test. Let $\sum_{n=m}^{\infty} a_n$ be a series of real numbers. Define $\alpha := \limsup_{n\to\infty} |a_n|^{1/n}$. (i) If $\alpha < 1$, then the series $\sum_{n=m}^{\infty} a_n$ is absolutely convergent. In particular, the series $\sum_{n=m}^{\infty} a_n$ is convergent. (ii) If $\alpha > 1$, then the series $\sum_{n=m}^{\infty} a_n$ is divergent. (iii) If $\alpha = 1$, no conclusion is asserted.

The Ratio Test. Let $\sum_{n=m}^{\infty} a_n$ be a series of nonzero numbers. (So, a_{n+1}/a_n is defined for any $n \geq m$.) (i) If $\limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$, then the series $\sum_{n=m}^{\infty} a_n$ is absolutely convergent. In particular, $\sum_{n=m}^{\infty} a_n$ is convergent. (ii) If $\liminf_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} > 1$, then the series $\sum_{n=m}^{\infty} a_n$ is divergent. In particular, $\sum_{n=m}^{\infty} a_n$ is not absolutely convergent.

Let X be a subset of $\mathbf{R}$ and let $f\colon X \to \mathbf{R}$ be a function. Let x_0 be an element of X. We say that f is **continuous** at x_0 if and only if

$$\lim_{x\to x_0; x\in X} f(x) = f(x_0).$$

That is, the limit of f at x_0 in X exists, and this limit is equal to $f(x_0)$. We say that f is **continuous on** X (or we just say that f is **continuous**) if and only if f is continuous at x_0 for every $x_0 \in X$. We say that f is **uniformly continuous** if and only if, for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $x \in X$ satisfies $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$. We say that f is **Lipschitz continuous** with constant L if and only if there exists $L \geq 0$ such that, for every $x, y \in X$, we have $|f(x) - f(y)| \leq L\,|x - y|$.

Intermediate Value Theorem. Let $a < b$ be real numbers. Let $f\colon [a, b] \to \mathbf{R}$ be function that is continuous on $[a, b]$. Let y be a real number between $f(a)$ and $f(b)$, so that either $f(a) \leq y \leq f(b)$ or $f(a) \geq y \geq f(b)$. Then there exists a $c \in [a, b]$ such that $f(c) = y$.

1. Label the following statements as TRUE or FALSE. If the statement is true, explain your reasoning. If the statement is false, provide a counterexample.

   (a) (2 points) Let $(a_n)_{n=0}^{\infty}$ be a convergent sequence of real numbers. Then $(a_n)_{n=0}^{\infty}$ is a Cauchy sequence.

   TRUE FALSE (circle one)

   (b) (2 points) Let $(a_n)_{n=0}^{\infty}$ be a bounded sequence of real numbers. Then $(a_n)_{n=0}^{\infty}$ is a convergent sequence.

   TRUE FALSE (circle one)

   (c) (3 points) Let $(a_n)_{n=0}^{\infty}$ be a positive, decreasing sequence of real numbers. (That is, $a_n \geq 0$ and $a_{n+1} \leq a_n$ for all $n \in \mathbf{N}$.) Then $\sum_{n=0}^{\infty}(-1)^n a_n$ converges.

   TRUE FALSE (circle one)

   (d) (3 points) Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers such that $\left|\frac{a_{n+1}}{a_n}\right| < 1$ for all natural numbers n. Then $\sum_{n=0}^{\infty} a_n$ converges.

   TRUE FALSE (circle one)

2. Determine which of the following series converges. Justify your answer

   (a) (5 points) $\sum_{n=1}^{\infty} \frac{n}{2^n}$.

   (b) (5 points) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

3. (10 points) Let $f\colon [0,1] \to [0,1]$ be a continuous function. Show that there exists some $x \in [0,1]$ such that $f(x) = x$. (Hint: apply the Intermediate Value Theorem to $g(x) := f(x) - x$.)

4. (10 points) Let $x > 1$. Prove that $\lim_{n \to \infty} \frac{x^n}{n} = +\infty$. (Hint: try writing $x = 1 + \varepsilon$ where $\varepsilon > 0$, then use the binomial theorem.)

5. (10 points) Let $x \in \mathbf{R}$. Consider the function $f(x) := \sum_{n=1}^{\infty} \frac{1}{x^2 + n^2}$. Prove that f is continuous on $(-\infty, +\infty)$. (Hint: it may be easier to prove that f is uniformly continuous on $\mathbf{R}$. You could start by trying to prove that the function $1/(x^2 + n^2)$ is Lipschitz continuous, for every $n \geq 1$.)

(Scratch paper)