131A Midterm 1 Solutions

1. Question 1

Prove the following statement: Let n be a positive integer. Then $1+2+\cdots+n = n(n+1)/2$.

Solution: We prove the assertion by induction on n. We first check the base case. In the case $n = 1$, note that $1(1+1)/2 = 2/2 = 1$, so the base case is verified. We now induct on n. Assume the assertion is true for a fixed positive integer n. We then prove the assertion in the case $n+1$. By the inductive hypothesis, $1+2+\cdots+n = n(n+1)/2$. Adding $n+1$ to both sides, we get

$$1+2+\cdots+n+(n+1) = n(n+1)/2+(n+1) = (n+1)((n/2)+1) = (n+1)(n+2)/2.$$

We have therefore completed the inductive step, and therefore proven the assertion for all positive integers n.

2. Question 2

(a) Define a Cauchy sequence of rational numbers.

Solution. $(a_n)_{n=0}^{\infty}$ is a Cauchy sequence of rational numbers if and only if $(a_n)_{n=0}^{\infty}$ is a sequence of rational numbers such that, for all rational $\varepsilon > 0$, there exists a natural number $N = N(\varepsilon)$ such that, for all $j, k \geq N$, we have $|a_j - a_k| < \varepsilon$.

(b) Let $(a_n)_{n=0}^{\infty}$ and let $(b_n)_{n=0}^{\infty}$ be Cauchy sequences of rational numbers. Prove that $(a_n + b_n)_{n=0}^{\infty}$ is a Cauchy sequence of rational numbers.

Solution. Let $\varepsilon > 0$ be rational. Note that $\varepsilon/2 > 0$ is also rational. So, applying the definition of a Cauchy sequence, there exist $L, M > 0$ such that, for all $j, k \geq L$, we have $|a_j - a_k| < \varepsilon/2$, and for all $j, k \geq M$ we have $|b_j - b_k| < \varepsilon/2$. Define $N := \max(L, M)$. Let $j, k \geq N$. Then, from the triangle inequality,

$$|a_j + b_j - a_k - b_k| = |a_j - a_k + b_j - b_k| \leq |a_j - a_k| + |b_j - b_k| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

That is, the sequence $(a_n + b_n)_{n=0}^{\infty}$ is a Cauchy sequence of rational numbers. (Note that since a_n and b_n are rational for every $n \in \mathbf{N}$, we know that $a_n + b_n$ is rational for all $n \in \mathbf{N}$.)

3. Question 3

Determine which of the following sequences converges. If the sequence converges, find its limit and prove that the sequence converges to this limit. If the sequence does not converge, prove it.

(a) Let $a_n = 1/n$ for any positive integer n.

Solution This sequence converges to 0. Let $\varepsilon > 0$ be a real number. Then $1/\varepsilon > 0$ is a real number. From a Proposition from class, there exists a natural number N such that $N > 1/\varepsilon > 0$, so that $0 < 1/N < \varepsilon$. Now, let $n \geq N$, so that $0 < 1/n \leq 1/N$. Then,

$$|a_n - 0| = |a_n| = |1/n| = 1/n \leq 1/N < \varepsilon.$$

That is, the sequence $(1/n)_{n=1}^{\infty}$ converges to 0, as desired.

(b) Let $a_n = (-1)^n$ for any positive integer n.

Solution There are many ways to prove this. Here are two ways.

Solution 1. Note that $|a_n - a_{n+1}| = 2$ for all positive integers n. So, the sequence $(a_n)_{n=1}^{\infty}$ cannot be Cauchy. So, this sequence cannot be convergent, since being a Cauchy sequence is equivalent to being a convergent sequence (from a Proposition from class/homework).

Solution 2. This sequence does not converge to any real number. We show this by contradiction. Assume that this sequence converges to a real number L. Then for all $\varepsilon > 0$, there exists a natural number $N = N(\varepsilon) > 0$ such that, for all $n \geq N$, we have $|a_n - L| < \varepsilon$. Choose for example $\varepsilon := 1/4 > 0$. Then, from the reverse triangle inequality, we have for all $n \geq N$

$$|a_n - L| = |a_n - a_{n+1} + a_{n+1} - L| \geq |a_n - a_{n+1}| - |a_{n+1} - L| = 2 - |a_{n+1} - L| > 2 - \varepsilon = 7/4.$$

That is, $|a_n - L| > 7/4$ for all $n \geq N$. This contradicts our convergence statement. We are therefore done.

4. Question 4

Let X and Y be countable, disjoint sets. Prove that $X \cup Y$ is a countable set. (Recall that a set A is countable if and only if, there exists a bijection $f: A \to \mathbf{N}$. That is, for every $n \in \mathbf{N}$ there exists exactly one $a \in A$ such that $f(a) = n$.)

Since X is countable, there exists $f: X \to \mathbf{N}$ a bijection. Since Y is countable, there exists $g: Y \to \mathbf{N}$ a bijection. Define $h: X \cup Y \to \mathbf{Z}$ by $h(x) := f(x)$ and $h(y) := -g(y) - 1$. We claim that h is a bijection. To see this, let $z \in \mathbf{Z}$. We want to find a unique $a \in X \cup Y$ such that $h(a) = z$. If $z \geq 0$, we claim that $h(f^{-1}(z)) = z$. If $z < 0$, we claim that $h(g^{-1}(-z-1)) = z$. In the case $z \geq 0$, note that $f^{-1}(z) \in X$, and by the definition of h, we have $h(f^{-1}(z)) = f(f^{-1}(z)) = z$, using that f is a bijection. Also, $f^{-1}(z)$ is the unique element of $X \cup Y$ that maps to z under h, since f is itself a bijection (so no other element of X maps to z), and by the definition of h, all elements of Y map to negative integers under h. In the case $z < 0$, note that $-z - 1 \geq 0$, and $g^{-1}(-z-1) \in Y$. And by the definition of h, we have $h(g^{-1}(-z-1)) = -g(g^{-1}(-z-1)) - 1 = -(-z-1) - 1 = z$, using that g is a bijection. Also, $g^{-1}(-z-1)$ is the unique element of $X \cup Y$ that maps to z under h, since g is itself a bijection (so no other element of Y maps to z), and by the definition of h, all elements of X map to natural numbers under h. In conclusion, h is a bijection. Since $\mathbf{Z}$ is countable (as shown in class), we conclude that $X \cup Y$ is countable.

5. Question 5

The following inductive argument is incorrect. Explain the flaw in the argument.

Claim: All horses on Earth are the same color.

Proof: We prove the claim by induction. Let k be a positive integer. In the case $k = 1$, a single horse has the same color as itself, so the case $k = 1$ of the induction is known. We now assume by induction that each set of k horses is of the same color. We want to show that a set of $k + 1$ horses is of the same color. Suppose I have a set C of $k + 1$ horses. If I remove one horse from this set of $k + 1$ horses, I have k horses of the same color, by the inductive hypothesis. Label this set of k horses as A. All horses in the set A are the same color.

Now, take the set of $k + 1$ horses and remove a different horse from this set than the one that we removed before. Label this new set of k horses as B. All horses in the set B are the same color, by the inductive hypothesis. Since A and B have some horses in common, the $(k+1)$ horses in C all must have the same color. We have therefore completed the induction, and the claim is proven. □

Solution. The assertion "Since A and B have some horses in common" is incorrect in the case $k = 1$. In this case, C consists of 2 horses, A consists of 1 horse, and B consists of 1 horse. And by the choice of A and B, we know that A and B are disjoint (in the case $k = 1$). Since A and B do not have any horse in common, they do not have to consist of the same color.