5: METRIC SPACES, TOPOLOGY, CONTINUITY, COMPACTNESS

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Abstract. These notes are mostly copied from those of T. Tao from 2003, available here

Contents

1. Introduction

In this course, we continue our study of analysis from your previous analysis course. Recall that you took an abstract approach to linear algebra by discussing linear transformations and vector spaces. Similarly, in this course, we will begin by presenting an abstract approach to analysis, which generalizes the things we did for analysis on the real line. Specifically, we will first generalize a lot of the arguments from the real line to the setting of metric spaces. We will then apply this general theory in our discussion of analysis on Euclidean spaces of any dimension, power series, and trigonometric functions. We will then discuss Fourier analysis, and the various intricacies of differentiation in Euclidean spaces.

2. Metric Spaces

Recall that a sequence of real numbers $(x_n)_{n=0}^{\infty}$ converges to a real number x if and only if, for every real $\varepsilon > 0$, there exists an integer $N = N(\varepsilon)$ such that, for all $n > N$, we have $|x_n - x| < \varepsilon$. That is, eventually, the sequence $(x_n)_{n=0}^{\infty}$ is within a distance ε of x. And this is true for any $\varepsilon > 0$. So, whenever we have a space of points, and we can define some notion of distance between two points, then we should be able to make a similar definition of convergence of sequences. We are therefore led to consider the following question. What are the crucial properties of the distance function $d\colon \mathbb{R} \times \mathbb{R} \to [0, \infty)$ where $d(x, y) := |x - y|$ that allow us to consider convergence of sequences? The following properties suffice, as we shall see further below.

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Definition 2.1 (Metric Space). A **metric space** (X, d) is a set X together with a function $d\colon X \times X \to [0, \infty)$ which satisfies the following properties.

- For all $x \in X$, we have $d(x, x) = 0$.
- For all $x, y \in X$ with $x \neq y$, we have $d(x, y) > 0$. (Positivity)
- For all $x, y \in X$, we have $d(x, y) = d(y, x)$. (Symmetry)
- For all $x, y, z \in X$, we have $d(x, z) \leq d(x, y) + d(y, z)$. (Triangle inequality)

Example 2.2 (The real line). As mentioned above, define the function $d\colon \mathbb{R} \times \mathbb{R} \to [0, \infty)$ by $d(x, y) := |x - y|$, where $x, y \in \mathbb{R}$. Then $(\mathbb{R}, d)$ is a metric space.

Example 2.3 (Euclidean space). Let n be a positive integer. Define $\mathbb{R}^n$ by

$$\mathbb{R}^n = \{(x_1, \ldots, x_n)\colon x_i \in \mathbb{R}\ \forall\, i \in \{1, \ldots, n\}\}.$$

Define the **Euclidean metric** (or ℓ_2 metric) $d\colon \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ by

$$d_{\ell_2}((x_1, \ldots, x_n), (y_1, \ldots, y_n)) := \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2}.$$

Then $(\mathbb{R}^n, d_{\ell_2})$ is a metric space.

There are actually many interesting metrics to consider on $\mathbb{R}^n$. Here is another one.

Example 2.4. Let n be a positive integer. Define the ℓ_1 metric $d_{\ell_1}\colon \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ by

$$d_{\ell_1}((x_1, \ldots, x_n), (y_1, \ldots, y_n)) := \sum_{i=1}^{n} |x_i - y_i|.$$

Then $(\mathbb{R}^n, d_{\ell_1})$ is a metric space.

The last two examples actually satisfy a few additional properties which you may recall from your linear algebra class.

Definition 2.5 (Normed linear space). Let X be a vector space over $\mathbb{R}$. A **normed linear space** $(X, \|\cdot\|)$ is a vector space X over $\mathbb{R}$ together with a norm function $\|\cdot\| : X \to [0, \infty)$ which satisfies the following properties.

- $\|0\| = 0$.
- For all $x \in X$ with $x \neq 0$, we have $\|x\| > 0$. (Positivity)
- For all $x \in X$ and for all $\alpha \in \mathbb{R}$, we $\|\alpha x\| = |\alpha|\,\|x\|$. (Homogeneity)
- For all $x, y \in X$, we have $\|x + y\| \leq \|x\| + \|y\|$. (Triangle inequality)

Exercise 2.6. Let $(X, \|\cdot\|)$ be a normed linear space. Define $d\colon X \times X \to \mathbb{R}$ by $d(x, y) := \|x - y\|$. Show that (X, d) is a metric space.

Example 2.7. Let n be a positive integer. Define the ℓ_2 norm on $\mathbb{R}^n$ by

$$\|(x_1, \ldots, x_n)\|_{\ell_2} := \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}.$$

Then $(\mathbb{R}^n, \|\cdot\|_{\ell_2})$ is a normed linear space. From Exercise 2.6, $(\mathbb{R}^n, d_{\ell_2})$ is a metric space, which we saw in Example 2.3.

Example 2.8. Let n be a positive integer. Define the ℓ_1 norm on $\mathbb{R}^n$ by

$$\|(x_1,\ldots,x_n)\|_{\ell_1} := \sum_{i=1}^{n} |x_i| .$$

Then $(\mathbb{R}^n, \|\cdot\|_{\ell_1})$ is a normed linear space. From Exercise 2.6, $(\mathbb{R}^n, d_{\ell_1})$ is a metric space, which we saw in Example 2.4.

Example 2.9. Let n be a positive integer. Define the ℓ_∞ norm on $\mathbb{R}^n$ by

$$\|(x_1,\ldots,x_n)\|_{\ell_\infty} := \max_{i=1,\ldots,n} |x_i| .$$

Then $(\mathbb{R}^n, \|\cdot\|_{\ell_\infty})$ is a normed linear space. From Exercise 2.6, $(\mathbb{R}^n, d_{\ell_\infty})$ is a metric space, where $d_{\ell_\infty}((x_1,\ldots,x_n),(y_1,\ldots,y_n)) := \max_{i=1}^{n} |x_i - y_i|$.

Example 2.10. Let n be a positive integer and let $1 \leq p < \infty$ be a real number. Define the ℓ_p norm on $\mathbb{R}^n$ by

$$\|(x_1,\ldots,x_n)\|_{\ell_p} := \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} .$$

Then $(\mathbb{R}^n, \|\cdot\|_{\ell_p})$ is a normed linear space, though the triangle inequality is a bit more difficult to prove. From Exercise 2.6, $(\mathbb{R}^n, d_{\ell_p})$ is a metric space.

Exercise 2.11. Let n be a positive integer and let $x \in \mathbb{R}^n$. Show that $\|x\|_{\ell_\infty} = \lim_{p\to\infty} \|x\|_{\ell_p}$.

Euclidean space is actually even more special than a normed linear space, which we also learned in linear algebra class. Specifically, $\mathbb{R}^n$ is an inner product space.

Definition 2.12 (Real Inner product space). Let X be a vector space over $\mathbb{R}$. A **real inner product space** $(X, \langle\cdot,\cdot\rangle)$ is a vector space X over $\mathbb{R}$ together with an inner product function $\langle\cdot,\cdot\rangle\colon X \times X \to \mathbb{R}$ which satisfies the following properties.

- $\langle 0, 0\rangle = 0$.
- For all $x \in X$ with $x \neq 0$, we have $\langle x, x\rangle > 0$.
- For all $x, y \in X$, we have $\langle x, y\rangle = \langle y, x\rangle$. (Symmetry)
- For all $x \in X$ and for all $\alpha \in \mathbb{R}$, we $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$. (Homogeneity)
- For all $x, y, z \in X$, we have $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$. (Linearity)

Exercise 2.13. Let $(X, \langle\cdot,\cdot\rangle)$ be a real inner product space. Define $\|\cdot\| : X \to [0,\infty)$ by $\|x\| := \sqrt{\langle x, x\rangle}$. Show that $(X, \|\cdot\|)$ is a normed linear space. Consequently, from Exercise 2.6, if we define $d\colon X \times X \to [0,\infty)$ by $d(x,y) := \sqrt{\langle (x-y),(x-y)\rangle}$, then (X, d) is a metric space.

In order to prove Exercise 2.13, the following inequality is useful.

Theorem 2.14 (Cauchy-Schwarz Inequality). *Let $(X, \langle\cdot,\cdot\rangle)$ be a real inner product space. Then, for all $x, y \in X$, we have*

$$|\langle x, y\rangle| \leq \langle x, x\rangle^{1/2}\langle y, y\rangle^{1/2}.$$

Proof. It follows from Definition 2.12 that, if $x = 0$, then $\langle x, y\rangle = 0$ for any $y \in X$. (You should have proven this in your linear algebra class.) So, if $x = 0$, then both sides of the Cauchy-Schwarz inequality are zero, and the inequality therefore holds. Similarly, if $y = 0$, then both sides of the inequality are zero. We therefore assume that $x \neq 0$ and $y \neq 0$. For any $x \in X$, define $\|x\| := \sqrt{\langle x, x\rangle}$.

Starting from x, we subtract the projection of x onto y. Define $\delta := -\langle x, y\rangle / \|y\|^2$. We then have

$$0 \leq \|x + \delta y\|^2 = \|x\|^2 + 2\delta\langle x, y\rangle + |\delta|^2 \|y\|^2 = \|x\|^2 - |\langle x, y\rangle|^2 / \|y\|^2 .$$

□

Remark 2.15. For any $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{R}^n$, define

$$\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n)\rangle := \sum_{i=1}^{n} x_i y_i.$$

Then $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ is an inner product space. Note that

$$\|(x_1, \ldots, x_n)\|_{\ell_2} = \langle (x_1, \ldots, x_n), (x_1, \ldots, x_n)\rangle^{1/2}.$$

However, there does not exist an inner product $\langle \cdot, \cdot \rangle'$ on $\mathbb{R}^n$ such that $\|(x_1, \ldots, x_n)\|_{\ell_1}$ is equal to $(\langle (x_1, \ldots, x_n), (x_1, \ldots, x_n)\rangle')^{1/2}$. The last statement is more difficult to prove.

In the first part of this course, we will mostly focus on metric spaces. Once we deal with Fourier series, the subject of inner product spaces will reappear. However, we will need to deal with complex inner product spaces, so we now recall their definition. Recall that, for $\alpha, \beta \in \mathbb{R}$ we define

$$\overline{\alpha + \beta\sqrt{-1}} := \alpha - \beta\sqrt{-1}.$$

Definition 2.16 (Complex Inner product space). Let X be a vector space over $\mathbb{C}$. A **complex inner product space** $(X, \langle \cdot, \cdot \rangle)$ is a vector space X together with an inner product function $\langle \cdot, \cdot \rangle \colon X \times X \to \mathbb{C}$ which satisfies the following properties.

- $\langle 0, 0\rangle = 0$.
- For all $x \in X$ with $x \neq 0$, we have $\langle x, x\rangle > 0$.
- For all $x, y \in X$, we have $\langle x, y\rangle = \overline{\langle y, x\rangle}$. (Conjugate Symmetry)
- For all $x \in X$ and for all $\alpha \in \mathbb{C}$, we $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$. (Homogeneity)
- For all $x, y, z \in X$, we have $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$. (Linearity)

Remark 2.17. Let $(X, \langle \cdot, \cdot \rangle)$ be a complex inner product space. Then, for any $x, y, z \in X$ and for any $\alpha \in \mathbb{C}$, it follows from Definition 2.16 that

- $\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle$.
- $\langle x, \alpha y\rangle = \overline{\alpha}\langle x, y\rangle$.

Example 2.18. Let n be a positive integer. Let $(z_1, \ldots, z_n), (w_1, \ldots, x_n) \in \mathbb{C}^n$. The standard inner product on $\mathbb{C}^n$ is defined by

$$\langle (z_1, \ldots, z_n), (w_1, \ldots, w_n)\rangle := \sum_{i=1}^{n} z_i \overline{w_i}.$$

Then $(\mathbb{C}^n, \langle \cdot, \cdot \rangle)$ is a complex inner product space.

Exercise 2.19 (Cauchy-Schwarz Inequality). Let $(X, \langle\cdot,\cdot\rangle)$ be a complex inner product space. Let $x, y \in X$. Modify the proof of Theorem 2.14 to prove the Cauchy-Schwarz inequality for complex inner product spaces:

$$|\langle x, y\rangle| \leq \langle x, x\rangle^{1/2}\langle y, y\rangle^{1/2}.$$

Exercise 2.20. Let $(X, \langle\cdot,\cdot\rangle)$ be a complex inner product space. Let $x, y \in X$. As usual, let $\|x\| := \sqrt{\langle x, x\rangle}$. Prove **Pythagoras's theorem**: if $\langle x, y\rangle = 0$, then $\|x + y\|^2 = \|x\|^2 + \|y\|^2$

2.1. **Convergence of Sequences.** We now define convergence of sequences in a metric space in a way which imitates the convergence of sequences of real numbers.

Definition 2.21. Let (X, d) be a metric space. Let $(x^{(j)})_{j=k}^{\infty}$ be a sequence of elements of X. Let x be an element of X. We say that the sequence $(x^{(j)})_{j=k}^{\infty}$ **converges to x with respect to the metric** d if and only if, for every $\varepsilon > 0$, there exists an integer $J = J(\varepsilon)$ such that, for all $j > J$, we have $d(x^{(j)}, x) < \varepsilon$.

Proposition 2.22. *Let n be a positive integer. Let $x \in \mathbb{R}^n$. Let $(x^{(j)})_{j=k}^{\infty}$ be a sequence of elements of $\mathbb{R}^n$. We write $x^{(j)} = (x_1^{(j)}, \ldots, x_n^{(j)})$, so that for each $1 \leq i \leq n$, we have $x_i^{(j)} \in \mathbb{R}$, that is, $x_i^{(j)}$ is the i^{th} coordinate of $x^{(j)}$. Then the following three statements are equivalent.*

- *$(x^{(j)})_{j=k}^{\infty}$ converges to x with respect to d_{ℓ_1}.*
- *$(x^{(j)})_{j=k}^{\infty}$ converges to x with respect to d_{ℓ_2}.*
- *$(x^{(j)})_{j=k}^{\infty}$ converges to x with respect to d_{ℓ_∞}.*

Exercise 2.23. Prove Proposition 2.22.

Due to Proposition 2.22, we say that the metrics d_{ℓ_1}, d_{ℓ_2} and d_{ℓ_∞} are equivalent on $\mathbb{R}^n$. In fact, for any $p, p' \in \mathbb{R}$ with $1 \leq p, p' \leq \infty$, the metrics d_{ℓ_p} and $d_{\ell_{p'}}$ are equivalent on $\mathbb{R}^n$. In fact, something stronger is true. Let $\|\cdot\|_a, \|\cdot\|_b$ be any two norms on $\mathbb{R}^n$. Define the metrics d_a, d_b so that, for any $x, y \in \mathbb{R}^n$, we have $d_a(x, y) := \|x - y\|_a$ and $d_b(x, y) := \|x - y\|_b$. Then d_a and d_b are equivalent on $\mathbb{R}^n$.

As in the case of the real line, a sequence cannot converge to two distinct points.

Proposition 2.24. *Let (X, d) be a metric space. Let $(x^{(j)})_{j=k}^{\infty}$ be a sequence of elements of X. Let x, x' be elements of X. Assume that the sequence $(x^{(j)})_{j=k}^{\infty}$ converges to x with respect to d. Assume also that the sequence $(x^{(j)})_{j=k}^{\infty}$ converges to x' with respect to d. Then $x = x'$.*

Exercise 2.25. Prove Proposition 2.24.

Due to Proposition 2.24, if $(x^{(j)})_{j=k}^{\infty}$ is a sequence of elements of X which converges to x with respect to d, we then write $\lim_{j\to\infty} x^{(j)} = x$. Although the latter notation has no indication of the metric d, confusion should often not arise as to what metric the convergence is using.

3. Topology of Metric Spaces

The open intervals such as $(0, 1)$ and closed intervals such as $[1, 2]$ played a central role in analysis on the real line. The open interval is a special case of an open set, and the closed interval is a special case of a closed set. There is a way to generalize the notions of open and closed set to general metric spaces, so we pursue these notions now. We begin by generalizing the notion of an open interval to the notion of a metric ball. The language of topology is used everywhere throughout mathematics, so it is quite useful even just to learn the terminology.

Definition 3.1 (Metric Ball). Let (X, d) be a metric space, let x_0 be a point in X, and let $r > 0$ be a positive real number. We define the **ball** $B_{(X,d)}(x_0, r)$ in X, centered at x_0 with radius r to be the set

$$B_{(X,d)}(x_0, r) := \{x \in X : d(x, x_0) < r\}.$$

When the metric space (X, d) is apparent, we abbreviate $B_{(X,d)}(x_0, r)$ as $B(x_0, r)$. Also, if $(X, \|\cdot\|)$ is a normed linear space, we write $B_{(X,\|\cdot\|)}(x_0, r)$ to denote the set $\{x \in X : \|x - x_0\| < r\}$.

Example 3.2. Let $x, y \in \mathbb{R}$. In $\mathbb{R}$ with the metric $d(x, y) := |x - y|$, note that $B_{(\mathbb{R},d)}(x_0, r)$ is the open interval $(x_0 - r, x_0 + r)$.

Example 3.3. In $\mathbb{R}^2$ with the metric d_{ℓ_2}, we have

$$B_{(\mathbb{R}^2,d_{\ell_2})}((0, 0), 1) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}.$$

However, with the metric d_{ℓ_1}, we have

$$B_{(\mathbb{R}^2,d_{\ell_1})}((0, 0), 1) = \{(x, y) \in \mathbb{R}^2 : |x| + |y| < 1\}.$$

Also, with the metric d_{ℓ_∞}, we have

$$B_{(\mathbb{R}^2,d_{\ell_\infty})}((0, 0), 1) = \{(x, y) \in \mathbb{R}^2 : \max\{|x|, |y|\} < 1\}.$$

So, $B_{(\mathbb{R}^2,d_{\ell_1})}((0, 0), 1)$ is a diamond, $B_{(\mathbb{R}^2,d_{\ell_2})}((0, 0), 1)$ is a disc, and $B_{(\mathbb{R}^2,d_{\ell_\infty})}((0, 0), 1)$ is a square. Moreover,

$$B_{(\mathbb{R}^2,d_{\ell_1})}((0, 0), 1) \subseteq B_{(\mathbb{R}^2,d_{\ell_2})}((0, 0), 1) \subseteq B_{(\mathbb{R}^2,d_{\ell_\infty})}((0, 0), 1).$$

Remark 3.4. Note that if (X, d) is a nonempty metric space, and if $x_0 \in X$ with $r > 0$, then $B_{(X,d)}(x_0, r)$ is nonempty, since it contains x_0. Moreover, if $0 < r < r'$, we have the containment $B_{(X,d)}(x_0, r) \subseteq B_{(X,d)}(x_0, r')$.

Definition 3.5. Let (X, d) be a metric space, let E be a subset of X, and let x_0 be a point in X. We say that x_0 is an **interior point** of E if and only if there exists $r > 0$ such that $B(x_0, r) \subseteq E$. We say that x_0 is an **exterior point** of E if and only if there exists $r > 0$ such that $B(x_0, r) \cap E = \emptyset$. We say that x_0 is a **boundary point** of E if and only if x_0 is neither an interior point nor an exterior point of E.

Remark 3.6. The set of all interior points of E is called the **interior** of E, and it is denoted as $\mathrm{int}(E)$. The set of all exterior points of E is called the **exterior** of E, and it is denoted as $\mathrm{ext}(E)$. The set of all boundary points of E is called the **boundary** of E, and it is denoted as ∂E.

Remark 3.7. If x_0 is an interior point of E, then x_0 is an element of E. If x_0 is an exterior point of E, then x_0 is not an element of E. If x_0 is a boundary point of E, then x_0 may or may not be an element of E.

Example 3.8. Consider the real line $\mathbb{R}$ with the usual metric. The open interval $(0,1)$ has interior $(0,1)$, it has exterior $(-\infty,0)\cup(1,\infty)$, and it has boundary $\{0,1\}$. The closed interval $[0,1]$ has interior $(0,1)$, it has exterior $(-\infty,0)\cup(1,\infty)$, and it has boundary $\{0,1\}$. The half-open interval $(0,1]$ has interior $(0,1)$, it has exterior $(-\infty,0)\cup(1,\infty)$, and it has boundary $\{0,1\}$.

Definition 3.9. Let (X,d) be a metric space, let E be a subset of X, and let x_0 be a point in X. We say that x_0 is an **adherent point** of E if and only if for every real $r>0$, we have $B(x_0,r)\cap E\neq\emptyset$. The set of all adherent points of E is called the **closure** of E and is denoted by $\overline{E}$.

The definitions of interior, exterior, boundary and closure are related by the following proposition.

Proposition 3.10. *Let (X,d) be a metric space, let E be a subset of X, and let x_0 be a point in X. Then the following statements are equivalent.*

- *x_0 is an adherent point of E.*
- *x_0 is either an interior point of E or a boundary point of E.*
- *There exists a sequence $(x^{(j)})_{j=1}^{\infty}$ of elements of E which converges to x_0 with respect to the metric d.*

Exercise 3.11. Prove Proposition 3.10.

We now define open and closed sets in terms of the boundary points of a set. As we will see below, we can equivalently define open and closed sets using only open balls.

Definition 3.12 (Open and Closed Sets). Let (X,d) be a metric space, let E be a subset of X. We say that E is **closed** if and only if E contains all of its boundary points, i.e. when $\partial E\subseteq E$. We say that E is **open** if and only if E contains none of its boundary points, i.e. when $(\partial E)\cap E=\emptyset$. If E contains some of its boundary points but not others, then E is neither open nor closed.

Remark 3.13. If a set E has no boundary, then E is simultaneously open and closed. For example, if $E=X$, then E is both open and closed. Also, if $E=\emptyset$, then E is both open and closed.

Example 3.14. We continue Example 3.8. Consider the real line $\mathbb{R}$ with the usual metric. The open interval $(0,1)$ has boundary $\{0,1\}$, so the open interval is open. The closed interval $[0,1]$ has boundary $\{0,1\}$, so the closed interval is closed. The half-open interval $(0,1]$ has boundary $\{0,1\}$, so the half-open interval is neither open nor closed.

As promised, we now show some equivalent definitions of open and closed sets.

Proposition 3.15 (Properties of Open and Closed Sets). *Let (X,d) be a metric space.*

(i) *Let E be a subset of X. Then E is open if and only if $E=\mathrm{int}(E)$. That is, E is open if and only if, for every $x\in E$, there exists an $r>0$ such that $B(x,r)\subseteq E$.*
(ii) *Let E be a subset of X. Then E is closed if and only if E contains all of its adherent points, i.e. when $E=\overline{E}$. That is, E is closed if and only if, for every convergent sequence $(x^{(j)})_{j=0}^{\infty}$ consisting of elements of E, the limit $\lim_{n\to\infty}x_n$ of the sequence also lies in E.*

(iii) *For any $x_0 \in X$, for any $r > 0$, the open ball $B(x_0, r)$ is an open set. The set $\{x \in X : d(x, x_0) \leq r\}$ is a closed set. The latter set is sometimes called the* ***closed ball*** *of radius r centered at x_0.*
(iv) *Let $x_0 \in X$. Then the singleton set $\{x_0\}$ is closed.*
(v) *If E is a subset of X, then E is open if and only if $X \smallsetminus E$ is closed. Here we have denoted $X \smallsetminus E := \{x \in X : x \notin E\}$ as the complement of E in X.*
(vi) *If $E_1, \ldots, E_n$ is a finite collection of open sets, then $E_1 \cap \cdots \cap E_n$ is an open set. If $F_1, \ldots, F_n$ is a finite collection of closed sets, then $F_1 \cup \cdots \cup F_n$ is a closed set.*
(vii) *If $\{E_\alpha\}_{\alpha \in I}$ is collection of open sets, (where the index set I can be finite, countable, or uncountable), then $\cup_{\alpha \in I} E_\alpha$ is an open set. If $\{F_\alpha\}_{\alpha \in I}$ is collection of closed sets, (where the index set I can be finite, countable, or uncountable), then $\cap_{\alpha \in I} F_\alpha$ is a closed set.*
(viii) *If E is any subset of X, then $\mathrm{int}(E)$ is the largest open set contained in E. That is, $\mathrm{int}(E)$ is open, and if V is any open set such that $V \subseteq E$, then $V \subseteq \mathrm{int}(E)$ also. Similarly, $\overline{E}$ is the smallest closed set containing E. That is, $\overline{E}$ is closed, and if V is any closed set such that $V \supseteq E$, then $V \supseteq \overline{E}$ also.*

Exercise 3.16. Prove Proposition 3.15.

Remark 3.17. Proposition 3.15(vi) does not hold for countable collections of sets, as we can see from the following examples which involve open and closed intervals on the real line.

$$\bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n+1}, 1 + \frac{1}{n+1}\right) = [0, 1].$$

$$\bigcup_{n \in \mathbb{N}} \left[\frac{1}{n+1}, 1 - \frac{1}{n+1}\right] = (0, 1).$$

As we see from the following example, it is natural to consider the open and closed sets of a subset of a metric space. Such notions are formalized by the relative topology.

Example 3.18. Consider $\mathbb{R}^2$, and define $Y := \{(x, 0) : x \in \mathbb{R}\}$, so that Y is a subset of $\mathbb{R}^2$. Let d_{ℓ_2} denote the ℓ_2 metric on $\mathbb{R}^2$. If we restrict d_{ℓ_2} to Y resulting in $d_{\ell_2}|_{Y \times Y}$, then $(Y, d_{\ell_2}|_{Y \times Y})$ is a metric space. In fact, we can identify $(Y, d_{\ell_2}|_{Y \times Y})$ with the real line $\mathbb{R}$ with its usual metric. Now, consider the set

$$E := \{(x, 0) : \; -1 < x < 1\}.$$

Then E is a subset of Y, and E is also a subset of $\mathbb{R}^2$. When we consider E as a subset of Y, then E is an open set, since E is equal to the ball $B_{(Y, d_{\ell_2}|_{Y \times Y})}((0, 0), 1)$. However, when we consider E as a subset of $\mathbb{R}^2$, then E is no longer an open set. To see this, note that for any $r > 0$, the ball $B_{(\mathbb{R}^2, d_{\ell_2})}((0, 0), r)$ is not contained in E. So, by Proposition 3.15(i), E is not open in $(\mathbb{R}^2, d_{\ell_2})$.

To summarize the above example: there is a sensible way to discuss open sets of a subset of a metric space, and it involves restricting the metric.

Definition 3.19 (**Relative Topology**). Let (X, d) be a metric space, let Y be a subset of X, and let E be a subset of Y. We say that E is **relatively open with respect to** Y if and only if E is open in the metric space $(Y, d|_{Y \times Y})$. Similarly, we say that E is **relatively closed with respect to** Y if and only if E is closed in the metric space $(Y, d|_{Y \times Y})$.

The definitions of relatively open and relatively closed sets are consistent with set intersection in the following way.

Proposition 3.20. *Let (X,d) be a metric space, let Y be a subset of X, and let E be a subset of Y.*

- *E is relatively open with respect to Y if and only if $E = V \cap Y$ for some set $V \subseteq X$ which is open in X.*
- *E is relatively closed with respect to Y if and only if $E = V \cap Y$ for some set $V \subseteq X$ which is closed in X.*

Proof. We only prove the first assertion, since the second assertion is proven similarly. First, assume that E is relatively open with respect to Y. Then E is open in the metric space $(Y, d|_{Y\times Y})$. So, by Proposition 3.15(i), for any $e \in E$ there exists $r = r(e) > 0$ such that $B_{(Y,d|_{Y\times Y})}(e, r(e)) \subseteq E$. Note that

$$E = \bigcup_{e\in E} B_{(Y,d|_{Y\times Y})}(e, r(e)). \tag{1}$$

Specifically, every set on the right is contained in E, so the union of the sets on the right is contained in E. And conversely, every element of E appears on the right side, so the right side contains the left side. We therefore deduce the equality (1). With this equality in mind, define

$$V := \bigcup_{e\in E} B_{(X,d)}(e, r(e)). \tag{2}$$

From Proposition 3.15(i), V is open in (X,d). For any $e \in E$, we have

$$Y \cap B_{(X,d)}(e, r(e)) = B_{(Y,d|_{Y\times Y})}(e, r(e)). \tag{3}$$

To see this, note that if $y \in B_{(Y,d|_{Y\times Y})}(e, r(e))$, then $d|_{Y\times Y}(e,y) < r(e)$. Since e and y are in Y, we have $d|_{Y\times Y}(e,y) = d(e,y)$. So, $d(e,y) < r(e)$, so that $y \in B_{(X,d)}(e, r(e))$. Conversely, let $y \in Y \cap B_{(X,d)}(e, r(e))$. then $d(e,y) < r(e)$. Since e and y are in Y, we again have $d(e,y) = d|_{Y\times Y}(e,y)$. So, $d|_{Y\times Y}(e,y) < r(e)$, so that $y \in B_{(Y,d|_{Y\times Y})}(e, r(e))$. In conclusion, (3) holds. Combining (1), (2) and (3), we conclude that

$$V \cap Y = Y \cap \left(\bigcup_{e\in E} B_{(X,d)}(e, r(e))\right) = \bigcup_{e\in E} \big(Y \cap B_{(X,d)}(e, r(e))\big) = \bigcup_{e\in E} B_{(Y,d|_{Y\times Y})}(e, r(e)) = E.$$

Conversely, assume that there exists $V \subseteq X$ which is open in X, such that $V \cap Y = E$. We will show that E is relatively open in Y. By Proposition 3.15(i), for any $v \in V$ there exists $r = r(v) > 0$ such that $B_{(X,d)}(e, r(e)) \subseteq V$. Note that

$$V = \bigcup_{v\in V} B_{(X,d)}(v, r(v)). \tag{4}$$

Now, consider the set

$$E' := \bigcup_{v\in V\cap Y} B_{(Y,d|_{Y\times Y})}(v, r(v)). \tag{5}$$

From Proposition 3.15(i), E' is open in $(Y, d|_{Y\times Y})$. We will then conclude by showing that $E = E'$. As before, for any $v \in V \cap Y$, we have the equality

$$Y \cap B_{(X,d)}(v, r(v)) = B_{(Y,d|_{Y\times Y})}(v, r(v)). \tag{6}$$

Combining (4), (5) and (6), we get $E = V \cap Y = E'$, completing the proof. $\square$

4. Cauchy sequences and Completeness

Recall the following definition of a subsequence. Let $(x^{(j)})_{j=k}^{\infty}$ be a sequence of elements of a metric space (X, d). Let $j_1, j_2, \ldots$ be an increasing sequence of integers such that

$$k \leq j_1 < j_2 < j_3 < \cdots$$

We then say that $(x^{(j_m)})_{m=1}^{\infty}$ is a **subsequence** of the sequence $(x^{(j)})_{j=k}^{\infty}$.

Lemma 4.1. *Let $(x^{(j)})_{j=k}^{\infty}$ be a sequence of elements of a metric space (X, d) which converges to some limit $x \in X$. Then every subsequence of $(x^{(j)})_{j=k}^{\infty}$ also converges to x.*

Exercise 4.2. Prove Lemma 4.1.

Definition 4.3 (Cauchy sequence). Let $(x^{(j)})_{j=k}^{\infty}$ be a sequence of elements of a metric space (X, d). We say that the sequence $(x^{(j)})_{j=k}^{\infty}$ is a **Cauchy sequence** if and only if, for every $\varepsilon > 0$, there exists an integer $J = J(\varepsilon)$ such that, for all $j, \ell > J$, we have $d(x^{(j)}, x^{(\ell)}) < \varepsilon$.

Lemma 4.4. *Let $(x^{(j)})_{j=k}^{\infty}$ be a sequence of elements of a metric space (X, d) which converges to some limit $x \in X$. Then $(x^{(j)})_{j=k}^{\infty}$ is also a Cauchy sequence.*

Exercise 4.5. Prove Lemma 4.1.

The converse is false sometimes, as we learned in the previous real analysis class. If $(x^{(j)})_{j=k}^{\infty}$ is a Cauchy sequence in X, then the sequence $(x^{(j)})_{j=k}^{\infty}$ may not converge to an element of X. For example, we saw that a Cauchy sequence of rationals numbers may converge to a real number which is not itself a rational number.

However, a Cauchy sequence of real numbers does converge to a real number, as we learned in the previous real analysis class.

Theorem 4.6. *Let $(\mathbb{R}, d)$ be the real line with the usual metric, so that for any $x, y \in \mathbb{R}$, we have $d(x, y) := |x - y|$. If $(x^{(j)})_{j=k}^{\infty}$ is a Cauchy sequence of elements of $\mathbb{R}$, then there exists some $x \in \mathbb{R}$ such that $(x^{(j)})_{j=k}^{\infty}$ converges to x with respect to d.*

We cast the latter property into the following definition.

Definition 4.7 (Completeness). Let (X, d) be a metric space. We say that (X, d) is **complete** if and only if the following property holds. For any Cauchy sequence $(x^{(j)})_{j=k}^{\infty}$ of elements of X, then there exists some $x \in X$ such that $(x^{(j)})_{j=k}^{\infty}$ converges to x with respect to d.

So, $(\mathbb{R}, d)$ is a complete metric space by Theorem 4.6. However, the metric space $(\mathbb{Q}, d)$ is not complete. For example, we can construct a sequence of rational numbers that converges to $\sqrt{2}$, but $\sqrt{2}$ is not a rational number.

Complete metric spaces are always closed when they are considered as subsets of other metric spaces, as we now show.

Proposition 4.8.

- *Let (X, d) be a metric space, and let Y be a subset of X, so that $(Y, d|_{Y \times Y})$ is a metric space. If $(Y, d|_{Y \times Y})$ is complete, then Y is closed in (X, d).*

- *Conversely, assume that (X,d) is a complete metric space and that Y is a closed subset of X. Then $(Y, d|_{Y\times Y})$ is complete.*

Exercise 4.9. Prove Proposition 4.8.

A metric space (X,d) which is not complete may or may not be closed, when considered as a subset of another metric space. For example, if d is the standard metric on $\mathbb{R}$, then $\mathbb{Q}$ is closed in $(\mathbb{Q}, d|_{\mathbb{Q}\times\mathbb{Q}})$, but $\mathbb{Q}$ is not closed in $(\mathbb{R}, d)$. We briefly mention that, given any metric space (X,d), there is a way to form the **completion** $(\overline{X}, \overline{d})$ of (X,d), so that $(\overline{X}, \overline{d})$ is a complete metric space that contains (X,d). This procedure imitates our construction of the real numbers using Cauchy sequences of rational numbers.

5. Compactness

We have now arrived at the extremely useful concept of compactness. Compactness expresses the exact properties that are needed to obtain the conclusion of the Bolzano-Weierstrass Theorem. As we recall, the Bolzano-Weierstrass theorem is very useful, and likewise compactness is very useful.

Definition 5.1 (**Boundedness**)**.** A sequence $(x^{(j)})_{j=k}^{\infty}$ in a metric space (X,d) is said to be **bounded** if and only if there exists $x \in X$ and there exists $r > 0$ such that $x^{(j)} \in B(x,r)$ for all $j \geq k$. Similarly, a subset E of a metric space (X,d) is said to be **bounded** if and only if there exists $x \in X$ and there exists $r > 0$ such that $E \subseteq B(x,r)$.

Theorem 5.2 (**Bolzano-Weierstrass**)**.** *Let $(\mathbb{R}, d)$ be the real line with the standard metric. Let $(x^{(j)})_{j=k}^{\infty}$ be a bounded sequence in $\mathbb{R}$. Then there exists a subsequence of $(x^{(j)})_{j=k}^{\infty}$ that converges in $(\mathbb{R}, d)$.*

Corollary 5.3. *Let n be a positive integer. Let $(\mathbb{R}^n, d)$ denote Euclidean space with either of the metrics $d = d_{\ell_2}$ or $d = d_{\ell_1}$. Let $(x^{(j)})_{j=k}^{\infty}$ be a bounded sequence in $\mathbb{R}^n$. Then there exists a subsequence of $(x^{(j)})_{j=k}^{\infty}$ that converges in $(\mathbb{R}^n, d)$.*

This convergent subsequence property is called compactness.

Definition 5.4 (**Compactness**)**.** A metric space (X,d) is said to be **compact** if and only if every sequence in (X,d) has at least one convergent subsequence.

A compact metric space satisfies the following two special properties.

Proposition 5.5. *Let (X,d) be a compact metric space. Then (X,d) is both complete and bounded.*

Exercise 5.6. Prove Proposition 5.5. (Hint: prove each property separately, and use argument by contradiction.)

We often talk about compact sets rather than compact metric spaces, so we make the following definition.

Definition 5.7 (**Compactness of a Set**)**.** Let (X,d) be a metric space, and let Y be a subset of X. We say that Y is **compact** if and only if the metric space $(Y, d|_{Y\times Y})$ is compact.

Corollary 5.8. *Let (X,d) be a metric space, and let Y be a compact subset of X. Then Y is closed and bounded.*

Proof. Apply Proposition 5.5 and then Proposition 4.8. □

In Euclidean space, the converse of Corollary 5.8 is true. The following Theorem therefore gives a useful characterization of compact subsets of Euclidean space.

Theorem 5.9. *Let n be a positive integer. Let $(\mathbb{R}^n, d)$ denote Euclidean space with the metric $d = d_{\ell_2}$ or $d = d_{\ell_1}$. Let E be a subset of $\mathbb{R}^n$. Then E is compact if and only if E is both closed and bounded.*

Exercise 5.10. Prove Theorem 5.9 using Corollary 5.3.

Compact sets of metric spaces can be equivalently characterized using open covers. This open cover property will become useful in our discussion of continuous functions. The property says: any (possibly uncountable) open cover of a compact set has a finite subcover. The following proof is a bit lengthy, so it can be skipped on a first reading.

Theorem 5.11 (Open Cover Characterization of Compactness). *Let (X, d) be a metric space and let Y be a compact subset of X. Let I be an index set. Let $\{V_\alpha\}_{\alpha\in I}$ be a collection of open sets in X. Assume that*

$$Y \subseteq \bigcup_{\alpha\in I} V_\alpha.$$

*(That is, the collection $\{V_\alpha\}_{\alpha\in I}$ **covers** Y.) Then, there exists a finite set $A \subseteq I$ such that*

$$Y \subseteq \bigcup_{\alpha\in A} V_\alpha.$$

Proof. Let $y \in Y$. Then there exists $\alpha \in I$ such that $y \in V_\alpha$. Since V_α is open, there exists $r > 0$ such that $B(y, r) \subseteq V_\alpha$. For each $y \in Y$, define $r(y) \in \mathbb{R}$ by

$$r(y) := \sup\{r \in (0, \infty) \colon \exists\, \alpha \in I \,\text{such that}\, B(y, r) \subseteq V_\alpha\}.$$

We showed that for every $y \in Y$, we have $r(y) > 0$. Define $r_0 \in \mathbb{R}$ by

$$r_0 := \inf\{r(y) \colon y \in Y\}.$$

Since $r(y) > 0$ for all $y \in Y$, we have $r_0 \geq 0$. We now consider the cases $r_0 = 0$ and $r_0 > 0$ separately.

Case 1. $r_0 = 0$. In this case, for every positive integer j, there exists $y \in Y$ such that $r(y) < 1/j$. So, for every positive integer j, let $y^{(j)} \in Y$ satisfy $r(y^{(j)}) < 1/j$. (We can do this by the countable axiom of choice.) By the Squeeze Theorem, $\lim_{j\to\infty} r(y^{(j)}) = 0$. Since $(y^{(j)})_{j=1}^\infty$ is a sequence in Y, and since Y is compact, there exists a subsequence $(y^{(j_k)})_{k=1}^\infty$ that converges to some point $y_0 \in Y$.

Since $y_0 \in Y$, we know as above that there exists $\alpha_0 \in I$ such that $y_0 \in V_{\alpha_0}$. And since V_{α_0} is open, there exists $\varepsilon > 0$ such that $B(y_0, \varepsilon) \subseteq V_{\alpha_0}$. Since $y^{(j_k)}$ converges to y_0 as $k \to \infty$, there exists a positive integer K such that, for all $k > K$, we have $y^{(j_k)} \in B(y_0, \varepsilon/2)$. By the triangle inequality, if $k > K$ and if $z \in B(y^{(j_k)}, \varepsilon/2)$, then $d(z, y_0) \leq d(z, y^{(j_k)}) + d(y^{(j_k)}, y_0) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. That is, if $k > K$, then $B(y^{(j_k)}, \varepsilon/2) \subseteq B(y_0, \varepsilon)$. Since $B(y_0, \varepsilon) \subseteq V_{\alpha_0}$, we conclude that $r(y^{(j_k)}) \geq \varepsilon/2$ for all $k > K$. The last condition implies that $\lim_{k\to\infty} r(y^{(j_k)}) \neq 0$, a contradiction. We conclude that Case 1 does not occur, i.e. we must have $r_0 > 0$.

Case 2. $r_0 > 0$. In this case, for every $y \in Y$, we have $r(y) > r_0/2$. So, for all $y \in Y$, there exists $\alpha = \alpha(y) \in I$ such that $B(y, r_0/2) \in V_\alpha$. We now argue by contradiction. Suppose there does not exist a finite collection $\{V_\alpha\}_{\alpha\in A}$ that covers Y. Let $y^{(1)}$ be any point in Y. We construct a sequence of points in Y recursively. Suppose we are given $y^{(1)},\ldots,y^{(j)}$ a sequence of points in Y. Given these points, the union $B(y^{(1)},r_0/2)\cup\cdots\cup B(y^{(j)},r_0/2)$ is contained in the union $V_{\alpha(y^{(1)})}\cup\cdots\cup V_{\alpha(y^{(j)})}$ for some $\alpha(y^{(1)}),\ldots,\alpha(y^{(j)})\in I$. By our contradictory assumption, the latter set does not cover Y, so the set $B(y^{(1)},r_0/2)\cup\cdots\cup B(y^{(j)},r_0/2)$ does not cover Y. That is, there exists some $y^{(j+1)}\in Y$ such that $y^{(j+1)}\notin B(y^{(i)},r_0/2)$ for all $1\leq i\leq j$. That is, $d(y^{(j+1)},y^{(i)})\geq r_0/2$ for all $1\leq i\leq j$. From the latter property, the sequence $(y^{(j)})_{j=1}^\infty$ is a sequence that has no convergent subsequence. (If a subsequence $(y^{(j_k)})_{k=1}^\infty$ converged to some $z\in Y$, then there would exist a positive integer K such that, for all $k>K$, we would have $d(y^{(j_k)},z)<r_0/4$, so that $d(y^{(j_{K+2})},y^{(j_{K+1})})\leq d(y^{(j_{K+2})},z)+d(z,y^{(j_{K+1})})<r_0/2$, a contradiction.) We have therefore contradicted the compactness of Y. Since we have achieved a contradiction, the proof is done. □

Remark 5.12. The converse is also true. If a set Y has the property that every open cover of Y has a finite subcover, then Y is compact.

Theorem 5.11 has the following useful corollary.

Corollary 5.13. *Let (X,d) be a metric space, and let $K_1,K_2,\ldots$ be a sequence of nonempty compact subsets of X such that*

$$K_1\supseteq K_2\supseteq K_3\supseteq\cdots.$$

Then the intersection $\cap_{j=1}^\infty K_j$ is nonempty.

Exercise 5.14. Prove Corollary 5.13. (Hint: first, work in the compact metric space $(K_1,d|_{K_1\times K_1})$. Then, consider the sets $K_1\smallsetminus K_j$ which are open in K_1. Assume for the sake of contradiction that $\cap_{j=1}^\infty K_j=\emptyset$. Then apply Theorem 5.11.)

Theorem 5.15. *Let (X,d) be a metric space.*

- (i) *Let Y be a compact subset of X, and let Z be a subset of Y. Then Z is compact if and only if Z is closed.*
- (ii) *Let $Y_1,\ldots,Y_n$ be compact subsets of X. Then $Y_1\cup\cdots\cup Y_n$ is compact.*
- (iii) *Every finite subset of X is compact.*

6. Continuity

We can readily generalize the notion of continuity of a function $f\colon\mathbb{R}\to\mathbb{R}$ to the setting of a function between metric spaces $f\colon X\to Y$. We just take the usual definition and then we replace the absolute values with the required metric, as follows.

Definition 6.1 (Continuity). Let (X,d_X) and let (Y,d_Y) be metric spaces. Let $f\colon X\to Y$ be a function. Let $x_0\in X$. We say that f is **continuous** at x_0 if and only if, for all $\varepsilon>0$, there exists $\delta=\delta(\varepsilon)>0$ such that, if $x\in X$ satisfies $d_X(x,x_0)<\delta$, then $d_Y(f(x),f(x_0))<\varepsilon$. We say that the function f is **continuous** if and only if it is continuous at every point in X.

Remark 6.2. Suppose $f\colon X\to Y$ is continuous and K is a subset of X. Then the restriction of f to K, $f|_K\colon K\to Y$ is also continuous.

As on the real line, continuous functions maps convergent sequences to convergent sequences.

Theorem 6.3. *Let (X, d_X) and let (Y, d_Y) be metric spaces. Let $f\colon X \to Y$ be a function. Then the following two statements are equivalent.*

- *f is continuous at x_0.*
- *If we have a sequence $(x^{(j)})_{j=1}^{\infty}$ in X which converges to x_0 with respect to d_X, then the sequence $(f(x^{(j)}))_{j=1}^{\infty}$ converges to $f(x_0)$ with respect to the metric d_Y.*

Exercise 6.4. Prove Theorem 6.3.

In fact, there is even a way to characterize continuous functions using the inverse images of open and closed sets.

Theorem 6.5. *Let (X, d_X) and let (Y, d_Y) be metric spaces. Let $f\colon X \to Y$ be a function. Then the following four statements are equivalent.*

- *f is continuous at x_0, for all $x_0 \in X$.*
- *For all $x_0 \in X$, if we have a sequence $(x^{(j)})_{j=1}^{\infty}$ in X which converges to x_0 with respect to d_X, then the sequence $(f(x^{(j)}))_{j=1}^{\infty}$ converges to $f(x_0)$ with respect to the metric d_Y.*
- *For all open sets W in Y, the set $f^{-1}(W) = \{x \in X : f(x) \in W\}$ is an open set in X.*
- *For all closed sets V in Y, the set $f^{-1}(V)$ is a closed set in X.*

Exercise 6.6. Prove Theorem 6.5.

Remark 6.7. For a continuous function, it is not always true that the image of an open set is open, and it is not always true that the image of a closed set is closed.

We can now quickly show that the composition of continuous functions is continuous.

Corollary 6.8. *Let (X, d_X), (Y, d_Y) and (Z, d_Z) be metric spaces. Let $f\colon (X, d_X) \to (Y, d_Y)$ be a continuous function and let $g\colon (Y, d_Y) \to (Z, d_Z)$ be a continuous function. Then $g \circ f\colon (X, d_X) \to (Z, d_Z)$ is a continuous function.*

Exercise 6.9. Prove Corollary 6.8.

7. Continuity and Compactness

Remark 7.1. From now on, unless otherwise specified, $\mathbb{R}^n$ refers to Euclidean space $\mathbb{R}^n$ with $n \geq 1$ a positive integer, and where we use the metric d_{ℓ_2} on $\mathbb{R}^n$. In particular, $\mathbb{R}$ refers to the metric space $\mathbb{R}$ equipped with the metric $d(x, y) = |x - y|$.

On the real line, we learned from the Extreme Value Theorem that the continuous image of a closed interval is another closed interval. The appropriate generalization of this statement to metric spaces now follows.

Theorem 7.2. *Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f\colon (X, d_X) \to (Y, d_Y)$ be a continuous function. Suppose $K \subseteq X$ is a compact set. Then $f(K) = \{f(x) : x \in K\}$ is also a compact set.*

Exercise 7.3. Prove Theorem 7.2

Combining this Theorem with the characterization of compactness in Euclidean spaces (i.e. Heine-Borel, Theorem 5.9), we get the following statement.

Corollary 7.4. *Let K be a closed and bounded subset of $\mathbb{R}^n$. Let $f: K \to \mathbb{R}^m$ be a continuous function. Then the set $f(K)$ is also closed and bounded. In particular, the function f is bounded on K.*

This Corollary allows us to state our generalization of the Extreme Value Theorem, which we now refer to as the Maximum Principle.

Definition 7.5. Let X be a set. Let $f: X \to \mathbb{R}$ be a function. We say that f **attains its maximum** at $x_0 \in X$ if and only if $f(x_0) \geq f(x)$ for all $x \in X$. We say that f **attains its minimum** at $x_0 \in X$ if and only if $f(x_0) \leq f(x)$ for all $x \in X$.

Theorem 7.6 (The Maximum Principle). *Let K be a closed and bounded subset of $\mathbb{R}^n$, and let $f: K \to \mathbb{R}$ be a continuous function. Then there exist points $a, b \in K$ such that f attains its maximum at a and f attains its minimum at b.*

Exercise 7.7. Prove Theorem 7.6. (Hint: use Corollary 7.4 and then consider the numbers $\sup_{x\in K} f(x)$ and $\inf_{x\in K} f(x)$.)

8. Continuity and Connectedness

Recall that the Intermediate Value Theorem says that a continuous function on an interval has an interval as its range. The appropriate generalization of this statement to metric spaces involves the concept of connectedness.

Definition 8.1 (Connectedness). Let (X, d) be a metric space. We say that X is **disconnected** if and only if there exist disjoint open sets V, W in X such that $V \cup W = X$. (Equivalently, X is disconnected if and only if X contains a proper non-empty subset which is both open and closed.) We say that X is **connected** if and only if X is not disconnected.

Example 8.2. The set $X = [0, 1] \cup [2, 3]$ with the metric $d(x, y) = |x - y|$ is disconnected, since the sets $[0, 1]$ and $[2, 3]$ are both open in X.

Definition 8.3. Let (X, d) be a metric space and let Y be a subset of X. We say that Y is **connected** if and only if the metric space $(Y, d|_{Y\times Y})$ is connected. We say that Y is **disconnected** if and only if the metric space $(Y, d|_{Y\times Y})$ is disconnected.

For the sake of examples, we now identify the connected subsets of the real line.

Theorem 8.4. *Let X be a subset of the real line $\mathbb{R}$. Then the following statements are equivalent.*

- *X is connected.*
- *For any $x, y \in X$ with $x < y$, the closed interval $[x, y]$ is also contained in X.*

Proof. We first show the forward implication. Suppose X is connected. We argue by contradiction. Let $x, y \in X$ with $x < y$ such that $[x, y]$ is not contained in X. Then there exists $x < z < y$ such that $z \notin X$. Then the sets $(-\infty, z) \cap X$ and $(z, \infty) \cap X$ are both disjoint, nonempty, relatively open sets whose union in X. Therefore, X is disconnected, a contradiction. We conclude that the forward implication holds.

We now prove the more involved reverse implication. Suppose for any $x, y \in X$ with $x < y$, the closed interval $[x, y]$ is also contained in X. We need to show that X is connected. We argue by contradiction. Suppose that X is disconnected. Then there exist two disjoint, nonempty, relatively open sets V, W such that $V \cup W = X$. Since V, W are nonempty, let $v \in V$ and let $w \in W$. Without loss of generality, $v < w$. By assumption, the closed interval $[v, w]$ is contained in X. Consider the real number

$$x = \sup([v, w] \cap V).$$

By the definition of x, we have $x \in [v, w]$. We will derive a contradiction by trying to determine whether or not $x \in V$.

Suppose $x \in V$. Since $w \notin V$, we have $x \neq w$, so $x \in [v, w)$. Since V is relatively open in X, there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \cap X \subseteq V$. Since $x \in [v, w) \subseteq X$ as well, there exists $\delta > 0$ such that $[x, x + \delta) \subseteq V$. But then x is not the least upper bound of $[v, w] \cap V$, a contradiction.

We must therefore have $x \notin V$. Since $x \in [v, w] \subseteq X$, and since V, W are disjoint, we must have $x \in W$. Since $v \in V$, we have $x \neq v$, so $x \in (v, w]$. Since W is relatively open in X, there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \cap X \subseteq W$. Since $x \in (v, w] \subseteq X$ as well, there exists $\delta > 0$ such that $(x - \delta, x] \subseteq W$. Since V and W are disjoint, we once again conclude that x is not the least upper bound of $[v, w] \cap V$. In any case, we have achieved a contradiction. We finally conclude that X is connected, as desired. □

Remark 8.5. So, $\mathbb{R}$ is connected, and so are the intervals $(a, b]$, $[a, b)$, (a, b), $[a, b]$, $(-\infty, b)$, $(-\infty, b]$, (a, ∞), $[a, \infty)$. Additionally, the empty set $\emptyset$ and singleton sets $\{a\}$ are connected. We therefore have a complete list of connected subsets of $\mathbb{R}$.

It turns out that connected sets are mapped to connected sets by continuous functions. This fact particularly implies the Intermediate Value Theorem.

Theorem 8.6. *Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f: X \to Y$ be a continuous function. Let E be a connected subset of X. Then $f(E)$ is connected.*

Exercise 8.7. Prove Theorem 8.6.

Theorem 8.8 (Intermediate Value Theorem). *Let (X, d) be a metric space. Let $f: X \to \mathbb{R}$ be a continuous function. Let E be a connected subset of X and let a, b be any two elements of E. Let y be a real number between $f(a)$ and $f(b)$, so that either $f(a) \leq y \leq f(b)$ or $f(b) \leq y \leq f(a)$. Then there exists $c \in E$ such that $f(c) = y$.*

Exercise 8.9. Prove Theorem 8.8 using Theorem 8.6.

9. Appendix: Notation

Let A, B be sets in a space X. Let m, n be a nonnegative integers.

$$\begin{aligned}
\mathbb{Z} &:= \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}, \text{ the integers}\\
\mathbb{N} &:= \{0, 1, 2, 3, 4, 5, \ldots\}, \text{ the natural numbers}\\
\mathbb{Z}_+ &:= \{1, 2, 3, 4, \ldots\}, \text{ the positive integers}\\
\mathbb{Q} &:= \{m/n \colon m, n \in \mathbb{Z}, n \neq 0\}, \text{ the rationals}\\
\mathbb{R} &\text{ denotes the set of real numbers}\\
\mathbb{R}^* &= \mathbb{R} \cup \{-\infty\} \cup \{+\infty\} \text{ denotes the set of extended real numbers}\\
\mathbb{C} &:= \{x + y\sqrt{-1} : x, y \in \mathbb{R}\}, \text{ the complex numbers}\\
\emptyset &\text{ denotes the empty set, the set consisting of zero elements}\\
\in &\text{ means ``is an element of.'' For example, } 2 \in \mathbb{Z} \text{ is read as ``2 is an element of } \mathbb{Z}.\text{''}\\
\forall &\text{ means ``for all''}\\
\exists &\text{ means ``there exists''}\\
\mathbb{R}^n &:= \{(x_1, \ldots, x_n) \colon x_i \in \mathbb{R}, \forall\, i \in \{1, \ldots, n\}\}\\
A \subseteq B &\text{ means } \forall\, a \in A, \text{ we have } a \in B, \text{ so } A \text{ is contained in } B\\
A \smallsetminus B &:= \{x \in A \colon x \notin B\}\\
A^c &:= X \smallsetminus A, \text{ the complement of } A\\
A \cap B &\text{ denotes the intersection of } A \text{ and } B\\
A \cup B &\text{ denotes the union of } A \text{ and } B
\end{aligned}$$

Let (X, d) be a metric space, let $x_0 \in X$, let $r > 0$ be a real number, and let E be a subset of X. Let $(x_1, \ldots, x_n)$ be an element of $\mathbb{R}^n$, and let $p \geq 1$ be a real number.

$$\begin{aligned}
B_{(X,d)}(x_0, r) = B(x_0, r) &:= \{x \in X \colon d(x, x_0) < r\}.\\
\overline{E} &\text{ denotes the closure of } E\\
\mathrm{int}(E) &\text{ denotes the interior of } E\\
\partial E &\text{ denotes the boundary of } E\\
\|(x_1, \ldots, x_n)\|_{\ell_p} &:= (\sum_{i=1}^{n} |x_i|^p)^{1/p}\\
\|(x_1, \ldots, x_n)\|_{\ell_\infty} &:= \max_{i=1,\ldots,n} |x_i|
\end{aligned}$$

Let $f \colon (X, d_X) \to (Y, d_Y)$ be a map between metric spaces. Let $V \subseteq X$, and let $W \subseteq Y$.

$$f(V) := \{f(v) \in Y \colon v \in V\}.$$

$$f^{-1}(W) := \{x \in X \colon f(x) \in W\}.$$

9.1. **Set Theory.** Let X, Y be sets, and let $f: X \to Y$ be a function. The function $f: X \to Y$ is said to be **injective** (or **one-to-one**) if and only if: for every $x, x' \in V$, if $f(x) = f(x')$, then $x = x'$.

The function $f: X \to Y$ is said to be **surjective** (or **onto**) if and only if: for every $y \in Y$, there exists $x \in X$ such that $f(x) = y$.

The function $f: X \to Y$ is said to be **bijective** (or a **one-to-one correspondence**) if and only if: for every $y \in Y$, there exists exactly one $x \in X$ such that $f(x) = y$. A function $f: X \to Y$ is bijective if and only if it is both injective and surjective.

Two sets X, Y are said to have the same **cardinality** if and only if there exists a bijection from X onto Y.

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