Please provide complete and well-written solutions to the following exercises.

Due April 5, at the beginning of class.

Homework 9

Exercise 1. Let $X_1, X_2, \ldots \colon \Omega \to \mathbf{R}$ be i.i.d. In each of the cases below, show that with probability one, $-\infty = \liminf_{n\to\infty} S_n$ and $\limsup_{n\to\infty} S_n = \infty$.

- The distribution μ_{X_1} is symmetric about 0 (i.e. $\mu_{-X_1} = \mu_{X_1}$) and $\mathbf{P}(X_1 = 0) < 1$.
- $\mathbf{E}X_1 = 0$ and $\mathbf{E}X_1^2 \in (0, \infty)$. (Hint: use the Central Limit Theorem.)

For example, when $\mathbf{P}(X_1 = 1) = \mathbf{P}(X_1 = -1) = 1/2$ and $S_0 = 0$, show that with probability one, $S_0, S_1, \ldots$ takes every integer value infinitely many times.

Exercise 2. Let M, N be stopping times for a random walk $S_0, S_1, \ldots$. Show that $\max(M, N)$ and $\min(M, N)$ are stopping times. In particular, if $n \geq 1$ is fixed, then $\max(M, n)$ and $\min(M, n)$ are stopping times

Exercise 3. Let $S_0, S_1, \ldots$ be a random walk with $S_0 = 0$. Let X be the number of times the random walk takes the value 0. Let $T := \min\{n \geq 1 \colon S_n = 0\}$.

- X is a geometric random variable with success probability $\mathbf{P}(T = \infty)$.
- $\mathbf{E}X = \frac{1}{\mathbf{P}(T=\infty)}$. (Here we interpret $1/0$ as ∞.)

(Hint: $\{X = k\} = \{T_{k-1} < \infty, T_k = \infty\} = \{T_{k-1} < \infty, T_k - T_{k-1} = \infty\}$.)

Exercise 4. Give a combinatorial proof that the simple random walk $S_0, S_1, \ldots$ on $\mathbf{Z}^d$ is recurrent for $d \leq 2$. That is, estimate $\mathbf{P}(S_n = 0) \approx n^{-d/2}$ when n is large and $d \leq 2$, and conclude $\sum_{n=0}^{\infty} \mathbf{P}(S_n = 0) = \infty$ for $d \leq 2$. (Hint: use Stirling's Formula.)

Exercise 5. Show that if the Simple Random Walk on $\mathbf{Z}^d$ is recurrent, then this random walk takes every value in $\mathbf{Z}^d$ infinitely many times. And if the Simple Random Walk on $\mathbf{Z}^d$ is transient, then this random walk takes any fixed value in $\mathbf{Z}^d$ only finitely many times.

Exercise 6. Let $0 < p < 1$. Consider the random walk on $\mathbf{Z}$ such that $\mathbf{P}(X_1 = 1) = p$ and $\mathbf{P}(X_1 = -1) = 1 - p$. Show that the corresponding random walk $S_0, S_1, \ldots$ is transient when $p \neq 1/2$.

Exercise 7. Let $S_0, S_1, \ldots$ and $S_0', S_1', \ldots$ be independent simple random walks on $\mathbf{Z}^d$. Let $N := \sum_{n,m\geq 0} 1_{S_n = S_m'}$ be the number of pairs of intersections of these two random walks. For any $y \in \mathbf{R}^d$, let $\phi(y) := \mathbf{E}e^{i\langle y, X_1\rangle}$.

- Show $\mathbf{E}N = \lim_{s\to 1^-} \int_{[-\pi,\pi]^d} \frac{1}{|1-s\phi(y)|^2} \frac{dy}{(2\pi)^d}$. (Hint: consider $\mathbf{E}e^{i\langle y, (S_n - S_m')\rangle}$.)
- For what $d \geq 1$ is $\mathbf{E}N < \infty$?

- Let $C := \{S_n \colon n \geq 0\} \cap \{S'_n \colon n \geq 0\}$ be the intersection set of the two independent random walks. Let $|C|$ denote the cardinality of C. Show that if the simple random walk on $\mathbf{Z}^d$ is transient, then $\mathbf{P}(N = \infty) = 1$ if and only if $\mathbf{P}(|C| = \infty) = 1$. (Hint: $N = \sum_{x \in C} N_x N'_x$ where $N_x := \sum_{n \geq 0} 1_{S_n = x}$ is the number of visits of the first random walk to x.) In the recurrent case $d = 1, 2$, Exercise 5 implies that $\mathbf{P}(|C| = \infty) = 1$. For any $d \geq 1$, note that $N < \infty$ implies $|C| < \infty$. It can also be shown that $\mathbf{P}(N < \infty) \in \{0, 1\}$, $\mathbf{P}(|C| = \infty) \in \{0, 1\}$, and that $\mathbf{P}(N < \infty) = 1$ if and only if $\mathbf{E}N < \infty$ (you don't have to show these things). In summary, $\mathbf{P}(|C| = \infty) = 1$ if and only if $\mathbf{E}N = \infty$.
- Hypothesize what happens to $\mathbf{E}N$ when we instead consider the tuples of intersections of $k > 2$ independent simple random walks in $\mathbf{R}^d$. (You don't have to prove your hypothesis.)

Exercise 8. Let $1/2 < p < 1$. Consider the random walk on $\mathbf{Z}$ such that $\mathbf{P}(X_1 = 1) = p$ and $\mathbf{P}(X_1 = -1) = 1 - p$. Let $S_0, S_1, \ldots$ be the corresponding random walk with $S_0 := 0$. Let $N := \min\{n \geq 1 \colon S_n > 0\}$. Using Wald's equation for $\min(N, n)$ and then letting $n \to \infty$, show that $\mathbf{E}N = 1/\mathbf{E}X_1 = 1/(2p - 1)$.