Please provide complete and well-written solutions to the following exercises.

Due January 25, at the beginning of class.

Homework 1

Exercise 1. This exercises gives a strategy for proving a property for a generated σ-algebra.

Let $\mathcal{A}$ be a collection of subsets of a set Ω, and let $p(A)$ be a property of subsets A of Ω, so that $p(A)$ is true or false for each $A \in \Omega$. Assume the following:

- $p(\emptyset)$ is true.
- $p(A)$ is true for all $A \in \mathcal{A}$.
- If $A \subseteq \Omega$ is such that $p(A)$ is true, then $p(A^c)$ is also true.
- If $A_1, A_2, \ldots \subseteq \Omega$ are such that $p(A_i)$ is true for all $i \geq 1$, then $p(\cup_{i=1}^{\infty} A_i)$ is true.

Show that $p(A)$ is true for all $A \in \sigma(\mathcal{A})$. (Hint: what can one say about $\{A \subseteq \Omega : p(A)\,\text{is true}\}$?

Exercise 2. Let $n \geq 1$. Show that the Borel σ-algebra on $\mathbf{R}^n$ is generated by sets of the form $A_1 \times \cdots \times A_n$ where $A_i \subseteq \mathbf{R}$ is a Borel set for every $1 \leq i \leq n$.

Exercise 3. Let $n, m \geq 1$. Let $f\colon \mathbf{R}^n \to \mathbf{R}^m$ be a continuous function. Show that f is measurable (if $\mathbf{R}^n, \mathbf{R}^m$ each have the Borel σ-algebra.)

Exercise 4. Let $X_1\colon \Omega \to S_1, \ldots, X_n\colon \Omega \to S_n$ be measurable functions. Show that the joint function $(X_1, \ldots, X_n)\colon \Omega \to S_1 \times \cdots \times S_n$ defined by

$$(X_1, \ldots, X_n)(\omega) := (X_1(\omega), \ldots, X_n(\omega)), \qquad \forall\, \omega \in \Omega$$

is measurable.

Exercise 5. Let μ be a measure on a measurable space $(\Omega, \mathcal{F})$. Using the axioms for a measure, show:

- (Monotonicity) If $A \subseteq B$ are measurable, then $\mu(A) \leq \mu(B)$.
- (Subadditivity) If $A_1, A_2, \ldots$ are measurable (but not necessarily disjoint), then
$$\mu(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu(A_n).$$
- (Continuity from below) If $A_1 \subseteq A_2 \subseteq \cdots$ are measurable, then $\mu(\cup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$.
- (Continuity from above) If $A_1 \supseteq A_2 \supseteq \cdots$ are measurable and if $\mu(A_1) < \infty$, then $\mu(\cap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$. Then, find a measurable space $(\Omega, \mathcal{F})$ and measurable subsets $B_1 \supseteq B_2 \supseteq \cdots$ such that $\mu(\cap_{n=1}^{\infty} B_n) \neq \lim_{n\to\infty} \mu(B_n)$.

Exercise 6. Let $(\Omega, \mathcal{F})$ be a measurable space. Let $[-\infty, \infty]$ have the Borel σ-algebra.

- Let $X\colon \Omega \to [-\infty, \infty]$. Show that X is measurable if and only if the sets $\{\omega \in \Omega\colon X(\omega) \leq t\}$ are measurable for all $t \in [-\infty, \infty]$.
- Let $X, Y\colon \Omega \to [-\infty, \infty]$. Show that $X = Y$ if and only if $\{\omega \in \Omega\colon X(\omega) \leq t\} = \{\omega \in \Omega\colon Y(\omega) \leq t\}$ for all $t \in [-\infty, \infty]$.
- Let $X_1, X_2, \ldots\colon \Omega \to [-\infty, \infty]$ be measurable. Show that $\sup_{m\geq 1} X_m, \inf_{m\geq 1} X_m$, $\limsup_{m\to\infty} X_m$, and $\liminf_{m\to\infty} X_m$ are all measurable.

Exercise 7. Let μ be a probability measure on $\mathbf{R}$, where $\mathbf{R}$ has the Borel σ-algebra. (Then μ is a Stieltjes measure.) Define the **distribution function** $F\colon \mathbf{R} \to [0, 1]$ associated to μ by

$$F(t) := \mu((-\infty, t]) = \mu(\{x \in \mathbf{R}\colon -\infty < x \leq t\}), \qquad \forall\, t \in \mathbf{R}.$$

Show the following properties of F:

- F is nondecreasing.
- $\lim_{t\to-\infty} F(t) = 0$ and $\lim_{t\to\infty} F(t) = 1$.
- F is right continuous, i.e. $F(t) = \lim_{s\to t^+} F(s)$ for all $t \in \mathbf{R}$.

Exercise 8. Let $F\colon \mathbf{R} \to [0, 1]$ satisfy the three properties from Exercise 7. Show that there exists a random variable X on $(0, 1)$ with the Borel σ-algebra such that

$$F(t) = \mathbf{P}(X \leq t), \qquad \forall\, t \in \mathbf{R}.$$

Here $\mathbf{P}$ is Lebesgue measure on $(0, 1)$. (Hint: consider $X(t) := \sup\{y \in \mathbf{R}\colon F(y) < t\}$. Then X is an inverse of F.)

Exercise 9. Let X be a random variable with cumulative distribution function $F\colon \mathbf{R} \to [0, 1]$. Show:

- $\mathbf{P}(X < t) = \lim_{s\to t^-} F(s)$.
- $\mathbf{P}(X = t) = F(t) - \lim_{s\to t^-} F(s)$.

So, $\mathbf{P}(X = t) = 0$ for all $t \in \mathbf{R}$ if and only if F is continuous.

Exercise 10. Let μ be a probability measure on $\mathbf{R}^n$, where $\mathbf{R}^n$ has the Borel σ-algebra. Define the **distribution function** $F\colon \mathbf{R}^n \to [0, 1]$ associated to μ by

$$\begin{aligned} F(t_1, \ldots, t_n) &:= \mu((-\infty, t_1] \times \cdots \times (-\infty, t_n]) \\ &= \mu(\{(x_1, \ldots, x_n) \in \mathbf{R}^n\colon -\infty < x_i \leq t_i, \forall\, 1 \leq i \leq n\}), \qquad \forall\, t_1, \ldots, t_n \in \mathbf{R}. \end{aligned}$$

Show the following properties of F:

- F is nondecreasing. ($F(t_1, \ldots, t_n) \leq F(t_1', \ldots, t_n')$ whenever $t_i \leq t_i'$ $\forall\ 1 \leq i \leq n$.)
- $\lim_{t_1,\ldots,t_n\to-\infty} F(t_1, \ldots, t_n) = 0$ and $\lim_{t_1,\ldots,t_n\to\infty} F(t_1, \ldots, t_n) = 1$.
- F is right continuous, i.e. $F(t_1, \ldots, t_n) = \lim_{(s_1,\ldots,s_n)\to(t_1,\ldots,t_n)^+} F(s_1, \ldots, t_n)$ for all $t_1, \ldots, t_n \in \mathbf{R}$, where the limit restricts that $s_i \geq t_i$ $\forall\ 1 \leq i \leq n$.
- If $t_{i,0} \leq t_{i,1}$ $\forall\ 1 \leq i \leq n$, then

$$\sum_{(\omega_1,\ldots,\omega_n)\in\{0,1\}^n} (-1)^{\omega_1+\cdots+\omega_n+n} F(t_{1,\omega_1}, \ldots, t_{n,\omega_n}) \geq 0.$$