Please provide complete and well-written solutions to the following exercises.

Due March 13, at the beginning of class.

Homework 4

Exercise 1. Let $\{Z_n\}_{n\in\mathbf{Z}}$ be a sequence of pairwise uncorrelated, mean zero random variables all with the same variance. Let a be 1 or -1. Show that there does not exist a weakly stationary time series $\{X_n\}_{n\in\mathbf{Z}}$ such that

$$X_n = aX_{n-1} + Z_n, \qquad \forall\, n \in \mathbf{Z}.$$

(Note that the associated function ϕ is $\phi(z) = 1 - az$ which has a zero on the unit circle $|z| = 1$.)

Exercise 2. Let p, q be positive integers. Let $a_1, \ldots, a_p, b_1, \ldots, b_q$ be real numbers. For any $z \in \mathbf{C}$, define $\phi(z) := 1-\sum_{j=1}^{p} a_j z^j, \qquad \theta(z) := 1+\sum_{j=1}^{q} b_j z^j$. Let $\{X_n\}_{n\in\mathbf{Z}}$ be a real-valued stochastic process. We define the shift operator S so that $SX_n := SX_{n-1}$, $\forall\, n \in \mathbf{Z}$. Let $\{Z_n\}_{n\in\mathbf{Z}}$ be a sequence of pairwise uncorrelated, mean zero random variables all with the same variance. Recall that the ARMA(p, q) process can be rewritten as

$$\phi(S)X_n = \theta(S)Z_n, \qquad \forall\, n \in \mathbf{Z}. \qquad (*)$$

(i) Suppose ϕ, θ have no common zeros and $\phi(z) \neq 0$ on $\{z \in \mathbf{C} \colon\ |z| = 1\}$. Let $\xi \colon \mathbf{C} \to \mathbf{C}$ be a polynomial that is nonzero on $\{z \in \mathbf{C} \colon\ |z| = 1\}$. If $\{X_n\}_{n\in\mathbf{Z}}$ is the unique weakly stationary solution to $(*)$, then $\{X_n\}_{n\in\mathbf{Z}}$ is also the unique weakly stationary solution to

$$\xi(S)\phi(S)X_n = \xi(S)\theta(S)Z_n, \qquad \forall\, n \in \mathbf{Z}.$$

(ii) Suppose ϕ, θ do have a common zero on the unit circle $\{z \in \mathbf{C} \colon\ |z| = 1\}$. Show that the equations $(*)$ can have more than one weakly stationary solution. (Hint: consider the equation $X_n = X_{n-1} + Z_n - Z_{n-1}$ $\forall\, n \in \mathbf{Z}$.)

Exercise 3. Let $\{Z_n\}_{n\in\mathbf{Z}}$ be a sequence of pairwise uncorrelated, mean zero random variables all with the same variance. Consider the equation

$$X_n = .4X_{n-1} + .21X_{n-2} + Z_n + .6Z_{n-1} + .09Z_{n-2}, \qquad \forall\, n \in \mathbf{Z}.$$

- Show that the associated polynomials ϕ, θ have a common root of $-10/3$.
- Recall that $\phi(S)X_n = \theta(S)Z_n$ for all $n \in \mathbf{Z}$. Define $\widetilde{\phi}(z) := \phi(z)/(1 + .3z)$, $\widetilde{\theta}(z) := \theta(z)/(1 + .3z)$, for all $z \in \mathbf{C}$. Show that

$$\widetilde{\phi}(S)X_n = \widetilde{\theta}(S)Z_n, \qquad \forall\, n \in \mathbf{Z}. \qquad (*)$$

- Deduce from the previous results that there exists a unique weakly stationary solution $\{X_n\}_{n\in\mathbf{Z}}$ to $(*)$. Moreover, the ARMA process is causal and invertible, and
$$X_n = Z_n + \sum_{j=1}^{\infty}(.7)^{j-1}Z_{n-j}, \qquad Z_n = X_n + \sum_{j=1}^{\infty}(-1)^j(.3)^{j-1}X_{n-j}, \qquad \forall\, n\in\mathbf{Z}.$$

Exercise 4. Let $\{X_n\}_{n\in\mathbf{Z}}$ be a weakly stationary, real-valued time series. For any $n\in\mathbf{Z}$, define the **difference operator** by
$$DX_n := X_n - X_{n-1}, \qquad \forall\, n\in\mathbf{Z}.$$
Define $D^1 := D$. And for any $k\geq 2$, define inductively
$$D^kX_n := D^{k-1}(DX_n), \qquad \forall\, n\in\mathbf{Z}.$$

- For any $k\geq 1$, is $\{D^kX_n\}_{n\in\mathbf{Z}}$ weakly stationary? If so, prove it.
- Let $\{a_n\}_{n\in\mathbf{Z}}$ be a sequence of real numbers with $\sum_{n\in\mathbf{Z}}|a_n|^2<\infty$. Let $f(x) := \sum_{n\in\mathbf{Z}} a_n e^{2\pi i n x}$ $\forall\, x\in\mathbf{R}/\mathbf{Z}$ be its Fourier transform. Recall that $a_n = \int_0^1 e^{-2\pi i n x}f(x)dx$ $\forall\, n\in\mathbf{Z}$. Write a similar integral expression for $D^k a_n$ for all $k\geq 1$, $n\in\mathbf{Z}$. That is, find $g_k\colon \mathbf{R}/\mathbf{Z}\to\mathbf{C}$ such that
$$D^k a_n = \int_0^1 g_k(x)e^{-2\pi i n x}f(x)dx, \qquad \forall\, k\geq 1,\ n\in\mathbf{Z}.$$
- What happens to g_k when k is large? For the sunspot data, examine D^kX_n for all $1\leq k\leq 5$, and examine the Fourier transforms. Are your observations consistent with the behavior of g_k when k is large?.