Please provide complete and well-written solutions to the following exercises.

Due February 28, at the beginning of class.

Homework 3

Exercise 1. Let H be a real Hilbert space. Verify the polarization identity:

$$4\langle h, g\rangle = \|h+g\|^2 - \|h-g\|^2 .$$

Now, assume that H is a complex Hilbert space. Verify the polarization identity

$$4\langle h, g\rangle = \sum_{j=0}^{3} i^j \left\|h + i^j g\right\|^2 .$$

Exercise 2. Let $\chi\colon \mathbf{R}/\mathbf{Z} \to \{z \in \mathbf{C}\colon |z| = 1\}$ be a character. (Recall: χ is a continuous function such that $\chi(s+t) = \chi(s)\chi(t)$ for all $s, t \in \mathbf{R}/Z$.)

- Show that $\chi'(0)$ exists.
- Show that $\chi'(t) = \chi(t)\chi'(0)$ for all $t \in \mathbf{R}/\mathbf{Z}$.
- Conclude that $\chi(t) = e^{2\pi i t y}$ for some $y \in \mathbf{R}$, and conclude that in fact $y \in \mathbf{Z}$.

Exercise 3 (Uniqueness of Fourier Series in L_1)**.** Let $f\colon \mathbf{R}/\mathbf{Z} \to \mathbf{C}$ with $\|f\|_1 := \int_0^1 |f(x)|\, dx < \infty$. Show: if $\widehat{f}(n) = 0$ for all $n \in \mathbf{Z}$, then $f = 0$. (Hint: show that $\lim_{N\to\infty} \|f * F_N - f\|_1 = 0$.) Since $\|f\|_1 \leq \|f\|$ by e.g. Jensen's inequality, conclude the uniqueness of Fourier series in $L_1(\mathbf{R}/\mathbf{Z}, \mathcal{B}, dx)$ as well.

Exercise 4. Let $f\colon \mathbf{R}/\mathbf{Z} \to \mathbf{C}$ with $\|f\|_1 := \int_0^1 |f(x)|\, dx < \infty$. In this Exercise, you are asked to show that

$$\lim_{n\to\pm\infty} \widehat{f}(n) = 0,$$

using the following plan.

- Show the assertion holds for any $f \in L_2(\mathbf{R}/\mathbf{Z}, \mathcal{B}, dx)$ as a consequence of Plancherel's Theorem. In particular, the assertion holds for any continuous f.
- Approximate f by a continuous $g\colon \mathbf{R}/\mathbf{Z} \to \mathbf{C}$. That is, for any $\varepsilon > 0$, find a continuous g satisfying $\|f - g\|_1 < \varepsilon$.
- Finally, use: if $\|f - g\|_1 < \varepsilon$, then $|\widehat{f}(n) - \widehat{g}(n)| < \varepsilon$ for all $n \in \mathbf{Z}$.

Show also that: if f is k times continuously differentiable, then there exists a constant c such that

$$|\widehat{f}(n)| \leq c n^{-k}.$$

Exercise 5 (Sunspot Data, Version 2)**.** This exercise deals with sunspot data from the following files (the same data appears in different formats) txt file csv (excel) file

These files are taken from http://www.sidc.be/silso/datafiles#total

To work with this data, e.g. in Matlab you can use the command

```
x=importdata('SN_d_tot_V2.0.txt')
```

to import the .txt file.

The format of the data is as follows.

- Columns 1-3: Gregorian calendar date (Year, Month, then Day)
- Column 4: Date in fraction of year
- Column 5: Daily total number of sunspots observed on the sun. A value of -1 indicates that no number is available for that day (missing value).
- Column 6: Daily standard deviation of the input sunspot numbers from individual stations.
- Column 7: Number of observations used to compute the daily value.
- Column 8: Definitive/provisional indicator. A blank indicates that the value is definitive. A '*' symbol indicates that the value is still provisional and is subject to a possible revision (Usually the last 3 to 6 months)

In a previous Exercise, we modelled the number of sunspots U_t at time t, where t is measured in years, by

$$U_t = m_t + a\cos(2\pi t/11) + b\sin(2\pi t/11) + Y_t, \qquad \forall\, t \in \mathbf{R},$$

where $a, b, \theta, \omega \in \mathbf{R}$ are unknown (deterministic) parameters, m_t is an unknown deterministic function of t that is assumed to be a "slowly varying" function of t, and $\{Y_t\}_{t\in\mathbf{R}}$ are i.i.d. mean zero random variables. The quantity m_t is called the **trend** and the quantity $s_t := a\cos(2\pi t/11) + b\sin(2\pi t/11)$ is called the **seasonal component** of the time series $\{U_t\}_{t\in\mathbf{R}}$.

This model was perhaps too simplistic, since it did not seem to fit the data well in some respects. This time, let's not make an a priori assumptions about known periodicities in the data. Consequently, we will just examine the Fourier coefficients of the data U_t directly. If we want to make plots of Fourier coefficients, it is easier to take absolute values. If the units of t were in integers, then we would define $\widehat{U}(s) = \sum_{t\in\mathbf{Z}} U_t e^{2\pi i s n}$ for any $s \in \mathbf{R}/\mathbf{Z}$, as the n^{th} frequency component of the time series. Since the units of t are in integers divided by 365 (or by 365.25), we instead define

$$\widehat{U}(r) := \sum_{t\in\mathbf{Z}/365} U_t e^{2\pi i t r}, \qquad \forall\, r \in \mathbf{R}/365\mathbf{Z}.$$

• Plot $|\widehat{U}(r)|$ versus r, where $r \in [0, 1]$ and also when $r \in [0, 365]$. Do you observe any large absolute values of $\widehat{U}(r)$ for any values of r near $1/11$?

You should observe some large values of $\widehat{U}(r)$ when r takes the values: $.0842, .0921$, and $.0995$, corresponding to frequencies of $11.87, 10.858$, and 10.05, respectively. This large signal should correspond to $r \in [.08, .105]$ (and to $r \in [-.105, -.08]$).

• Plot the inverse Fourier transform of this part of the frequency spectrum $\widehat{U}(r)$. That is, plot the following function of $t \in \mathbf{Z}/365$

$$S_t := \frac{1}{365} \int_{.08}^{.105} \widehat{U}(r) e^{-2\pi i r t} dr + \frac{1}{365} \int_{-.105}^{-.08} \widehat{U}(r) e^{-2\pi i r t} dr.$$

(Since the time series is real valued, $\widehat{U}(r) = \overline{\widehat{U}(-r)}$ for all $r \in \mathbf{R}/\mathbf{Z}$. Also, for any $x, y \in \mathbf{R}$, $\mathrm{Re}(x + y\sqrt{-1}) := x$.)

How does S_t compare to U_t when you put them in the same plot? (Instead of plotting U_t itself for this comparison, consider plotting a moving average of U_t.)

When you plotted S_t versus U_t, S_t should follow the oscillations of U_t fairly closely.

When you plotted $\widehat{U}(r)$ versus r, you should have also noticed large values for r near 0. These low frequencies correspond to the long term "trend" in the data. The low frequency signal should correspond roughly to $r \in [0, .016]$

• Plot the inverse Fourier transform of this part of the frequency spectrum $\widehat{U}(r)$. That is, plot the following function of $t \in \mathbf{Z}/365$

$$M_t := \frac{1}{365} \int_{-.016}^{.016} \widehat{U}(r) e^{-2\pi i r t} dr.$$

Plot U_t, M_t and S_t in the same plot. Then plot $U_t - S_t - M_t$. Does $U_t - S_t - M_t$ "resemble" a stationary process? Is this procedure better or worse than what we did on the previous homework?

• If $\{U_t - S_t - M_t\}_{t \in \mathbf{Z}/365}$ were a sequence of i.i.d. random variables with mean zero and variance one, what would its Fourier transform look like? That is, if $\{Z_t\}_{t \in \mathbf{Z}/365}$ were a sequence of i.i.d. random variables with mean zero and variance one, and if

$$\widehat{Z}(r) := \sum_{t \in \mathbf{Z}/365} Z_t e^{2\pi i t r}, \qquad \forall\, r \in \mathbf{R}/365\mathbf{Z},$$

then what would this function look like? Does it have mean zero (when r is fixed)? Can you compute the variance of $\widehat{Z}(r)$ (when r is fixed)? Are the quantities $\widehat{Z}(r)$ and $\widehat{Z}(s)$ independent when $s \neq r$, $s, r \in \mathbf{R}/365\mathbf{Z}$? If they are not independent, could you compute their covariance?