Please provide complete and well-written solutions to the following exercises.

Due January 24, at the beginning of class.

Homework 1

Exercise 1. Let A, B be events in a sample space. Let $C_1, \ldots, C_n$ be events such that $C_i \cap C_j = \emptyset$ for any $i, j \in \{1, \ldots, n\}$ with $i \neq j$, and such that $\cup_{i=1}^n C_i$ is the whole sample space. Show:

$$\mathbf{P}(A|B) = \sum_{i=1}^{n} \mathbf{P}(A|B,\, C_i)\mathbf{P}(C_i|B).$$

(Hint: consider using the Total Probability Theorem and that $\mathbf{P}(\cdot|B)$ is a probability law.)

Exercise 2. By definition, a random vector $Z = (Z_1, \ldots, Z_d) \in \mathbf{R}^d$ is **Gaussian** if, for any $v_1, \ldots, v_d \in \mathbf{R}$, the random variable $\sum_{i=1}^d v_i Z_i$ is a Gaussian random variable. Equivalently, for any $v \in \mathbf{R}^d$, the random variable $\langle v, Z \rangle$ is a Gaussian random variable. The covariance matrix $(a_{ij})_{1 \leq i,j \leq d}$ of Z is defined by

$$a_{ij} := \mathbf{E}((Z_i - \mathbf{E}Z_i)(Z_j - \mathbf{E}Z_j)).$$

Let $Z = (Z_1, \ldots, Z_d) \in \mathbf{R}^d$ be a Gaussian random vector.

- Show that the covariance matrix $(a_{ij})_{1 \leq i,j \leq d}$ of Z is symmetric, positive semidefinite. That is, for any $v \in \mathbf{R}^d$, we have

$$v^T a v = \sum_{i,j=1}^{d} v_i v_j a_{ij} \geq 0.$$

- Given any symmetric positive semidefinite matrix $(b_{ij})_{1 \leq i,j \leq d}$, show that there exists a Gaussian random vector Z such that the covariance matrix of Z is $(b_{ij})_{1 \leq i,j \leq d}$. (Hint: write the matrix b in its Cholesky decomposition $b = rr^*$, where r is a $d \times d$ real matrix. Let $e^{(1)}, \ldots, e^{(d)}$ be the rows of r. Let $X_1, \ldots, X_d$ be independent standard Gaussian random variables. Let $X := (X_1, \ldots, X_d)$. Define $Z_i := \langle X, e^{(i)} \rangle$ for any $1 \leq i \leq d$.)

Exercise 3. Let $Y_0, Y_1, \ldots$ be independent standard Gaussian random variables (so that they each have mean zero and variance one). Let $a, b \in \mathbf{R}$ be unknown (deterministic) parameters. For any $n \geq 0$, define

$$X_n := a + bY_n.$$

Suppose the data $X_0, X_1, \ldots, X_{30}$ are given by the following

$$0.7118\ 0.7587\ 0.9143\ -0.3666\ 2.2630\ 1.5951\ 0.9470\ 2.2222\ 0.3731\ 1.3387\ 1.2551\ 1.5915$$

$$1.5877\ 0.8811\ 1.3820\ 1.3011\ 1.7766\ 2.0560\ 2.0656\ 0.8818\ 1.4464\ 0.6715\ 0.7319\ 1.3959$$

$$2.3196\ 0.9382\ 1.6228\ 1.2646\ 2.0704\ 0.7466\ 1.4195$$

To the best of your ability, estimate a and b.

Exercise 4. Let μ be a probability measure on $\mathbf{R}^n$, where $\mathbf{R}^n$ has the Borel σ-algebra. Define the **distribution function** $F\colon \mathbf{R}^n \to [0,1]$ associated to μ by

$$\begin{aligned} F(t_1,\ldots,t_n) &:= \mu((-\infty,t_1]\times\cdots\times(-\infty,t_n]) \\ &= \mu(\{(x_1,\ldots,x_n)\in\mathbf{R}^n\colon -\infty < x_i \leq t_i,\, \forall\, 1\leq i\leq n\}), \qquad \forall\, t_1,\ldots,t_n\in\mathbf{R}. \end{aligned}$$

Show the following properties of F:

- F is nondecreasing. ($F(t_1,\ldots,t_n)\leq F(t_1',\ldots,t_n')$ whenever $t_i\leq t_i'$ $\forall\ 1\leq i\leq n$.)
- $\lim_{t_1,\ldots,t_n\to-\infty} F(t_1,\ldots,t_n)=0$ and $\lim_{t_1,\ldots,t_n\to\infty} F(t_1,\ldots,t_n)=1$.
- F is right continuous, i.e. $F(t_1,\ldots,t_n)=\lim_{(s_1,\ldots,s_n)\to(t_1,\ldots,t_n)^+} F(s_1,\ldots,t_n)$ for all $t_1,\ldots,t_n\in\mathbf{R}$, where the limit restricts that $s_i\geq t_i$ $\forall\ 1\leq i\leq n$.
- If $t_{i,0}\leq t_{i,1}$ $\forall\ 1\leq i\leq n$, then

$$\sum_{(\omega_1,\ldots,\omega_n)\in\{0,1\}^n} (-1)^{\omega_1+\cdots+\omega_n+n} F(t_{1,\omega_1},\ldots,t_{n,\omega_n}) \geq 0.$$

Exercise 5. A **finite Markov Chain** is a stochastic process $(X_0,X_1,X_2,\ldots)$ together with a finite set Ω, which is called the **state space** of the Markov Chain, and an $|\Omega|\times|\Omega|$ real matrix P. The random variables $X_0,X_1,\ldots$ take values in the finite set Ω. The matrix P is **stochastic**, that is all of its entries are nonnegative and

$$\sum_{y\in\Omega} P(x,y)=1, \qquad \forall\, x\in\Omega.$$

And the stochastic process satisfies the following **Markov property**: for all $x,y\in\Omega$, for any $n\geq 1$, and for all events H_{n-1} of the form $H_{n-1}=\cap_{k=0}^{n-1}\{X_k=x_k\}$, where $x_k\in\Omega$ for all $0\leq k\leq n-1$, such that $\mathbf{P}(H_{n-1}\cap\{X_n=x\})>0$, we have

$$\mathbf{P}(X_{n+1}=y\,|\,H_{n-1}\cap\{X_n=x\})=\mathbf{P}(X_{n+1}=y\,|\,X_n=x)=P(x,y).$$

That is, the next location of the Markov chain only depends on its current location. And the transition probability is defined by $P(x,y)$.

Suppose we have a Markov Chain with state space Ω. Let $n\geq 0$, $\ell\geq 1$, let $x_0,\ldots,x_n\in\Omega$ and let $A\subseteq\Omega^\ell$. Using the (usual) Markov property, show that

$$\begin{aligned} &\mathbf{P}((X_{n+1},\ldots,X_{n+\ell})\in A\,|\,(X_0,\ldots,X_n)=(x_0,\ldots,x_n)) \\ &\qquad = \mathbf{P}((X_{n+1},\ldots,X_{n+\ell})\in A\,|\,X_n=x_n). \end{aligned}$$

Then, show that

$$\mathbf{P}((X_{n+1},\ldots,X_{n+\ell})\in A\,|\,X_n=x_n)=\mathbf{P}((X_1,\ldots,X_\ell)\in A\,|\,X_0=x_n).$$

(Hint: it may be helpful to use the Multiplication Rule for conditional probabilities.)

Exercise 6. Let P,Q be stochastic matrices of the same size. Show that PQ is a stochastic matrix. Conclude that, if r is a positive integer, then P^r is a stochastic matrix.

Exercise 7. Let $Y_1,Y_2,\ldots\colon\Omega\to\mathbf{R}$ be random variables that converge almost surely to a random variable $Y\colon\Omega\to\mathbf{R}$. Show that $Y_1,Y_2,\ldots$ converges in probability to Y in the following way.

- For any $\varepsilon > 0$ and for any positive integer n, let
$$A_{n,\varepsilon} := \bigcup_{m=n}^{\infty} \{\omega \in \Omega\colon\ |Y_m(\omega) - Y(\omega)| > \varepsilon\}.$$
Show that $A_{n,\varepsilon} \supseteq A_{n+1,\varepsilon} \supseteq A_{n+2,\varepsilon} \supseteq \cdots$.
- Show that $\mathbf{P}(\cap_{n=1}^{\infty} A_{n,\varepsilon}) = 0$.
- Using Continuity of the Probability Law, deduce that $\lim_{n\to\infty} \mathbf{P}(A_{n,\varepsilon}) = 0$.

Now, show that the converse is false. That is, find random variables $Y_1, Y_2, \ldots$ that converge in probability to Y, but where $Y_1, Y_2, \ldots$ do not converge to Y almost surely.

Exercise 8. Suppose random variables $Y_1, Y_2, \ldots\colon \Omega \to \mathbf{R}$ converge in probability to a random variable $Y\colon \Omega \to \mathbf{R}$. Prove that $Y_1, Y_2, \ldots$ converge in distribution to Y.

Then, show that the converse is false.