545 Midterm 1 Solutions[1]

1. Question 1

Find a sequence of functions $f_1, f_2, \ldots : [0,1] \to \mathbf{R}$ such that

$$\lim_{n\to\infty} \int_0^1 |f_n(x)|^2\, dx = \infty,$$

and such that

$$\forall\, x \in [0,1], \quad \lim_{n\to\infty} f_n(x) = 0.$$

Prove your assertion. (In particular, you are asked to find a sequence of functions that converges pointwise to 0, such that this sequence does not converge to 0 in L_2.)

Solution. Let $f_n := n1_{(0,1/n]}$. The second item follows by definition, since $f_n(0) = 0$ for all $n \geq 1$ and for any $x \in (0,1]$, $f_n(x) = 0$ for all $n > 1/x$. The first item follows by direct calculation, since $\int_0^1 |f_n(x)|^2\, dx = n^2/n = n$.

2. Question 2

Let X, Y be real-valued random variables with $\mathbf{E}\,|X| < \infty$. Let $f\colon \mathbf{R} \to \mathbf{R}$ be a bounded (measurable) function. Using the definition of conditional expectation, show that

$$\mathbf{E}(Xf(Y)\,|\,Y) = f(Y)\mathbf{E}(X|Y).$$

Solution. Let $W := \mathbf{E}(Xf(Y)\,|\,Y)$. Let $Z := \mathbf{E}(X|Y)$. We are required to show that $W = f(Y)Z$. By the definition of W (i.e. the definition of conditional expectation), we are required to show that: for any bounded (measurable) $g\colon \mathbf{R} \to \mathbf{R}$, we have

$$\mathbf{E}Xf(Y)g(Y) = \mathbf{E}f(Y)Zg(Y)$$

By definition of conditional expectation $\mathbf{E}(X|Y)$, for any bounded (measurable) function $h\colon \mathbf{R} \to \mathbf{R}$, we have

$$\mathbf{E}Zh(Y) = \mathbf{E}Xh(Y). \qquad (**)$$

Choose $h := f \cdot g$. Then by definition of h,

$$\mathbf{E}Zf(Y)g(Y) = \mathbf{E}Zh(Y) \overset{(**)}{=} \mathbf{E}Xh(Y) = \mathbf{E}Xf(Y)g(Y)$$

3. Question 3

Let $f(x) := x$ for all $x \in [-1/2, 1/2]$. Define $\widehat{f}(n) := \int_{-1/2}^{1/2} f(x)e^{-2\pi i n x}dx$ for all $n \in \mathbf{Z}$. Show that

$$\widehat{f}(n) = \frac{(-1)^{n+1}}{2\pi i n}, \qquad \forall\, n \in \mathbf{Z},\, n \neq 0.$$

Conclude that

$$\sum_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

[1]

(Hint: use Parseval's Identity.) *Solution.* Integrating by parts for any $n \neq 0$: $\int_{-1/2}^{1/2} xe^{-2\pi inx}dx = \frac{-1}{2\pi in}\int_{-1/2}^{1/2} x(d/dx)e^{-2\pi inx}dx = \frac{-1}{2\pi in}[(1/2)e^{-\pi in} + (1/2)e^{\pi in}] = \frac{-1}{2\pi in}\cos(\pi n) = \frac{-1}{2\pi in}(-1)^{n+1}$. From Parseval's Theorem, we therefore have

$$\frac{1}{12} = \frac{2}{3}2^{-3} = \int_{-1/2}^{1/2} x^2 dx = \int_{-1/2}^{1/2} |f(x)|^2\, dx = \sum_{n\in\mathbf{Z},\, n\neq 0} |\widehat{f}(n)|^2 = \sum_{n\in\mathbf{Z},\, n\neq 0} \frac{1}{4\pi^2 n^2} = \sum_{n\geq 1} \frac{1}{2\pi^2 n^2}.$$

That is,

$$\sum_{n\geq 1} \frac{1}{n^2} = \frac{2}{12}\pi^2 = \frac{\pi^2}{6}.$$

4. Question 4

Let H be a real n-dimensional inner product space over $\mathbf{R}$. Let K be a real n-dimensional inner product space over $\mathbf{R}$. As we discussed in class, H and K are linearly isometric, in the sense that: $\exists$ a linear function $f\colon H \to K$ such that

$$\langle f(g), f(h)\rangle = \langle g, h\rangle, \qquad \forall\, g, h \in H,$$

and such that f is injective (if $f(h) = 0$, then $h = 0$).

Extend this result to the infinite-dimensional setting. That is, let H, K be real, separable, infinite-dimensional Hilbert spaces over $\mathbf{R}$. (An infinite-dimensional Hilbert space H is separable if there exists a countable orthonormal basis $\{h_n\}_{n\geq 0} \subseteq H$, i.e. $\langle h_n, h_m\rangle = 1_{\{n=m\}}$ for all $n, m \geq 0$, and for any $h \in H$, $\lim_{m\to\infty} \|h - \sum_{n=0}^{m}\langle h, h_n\rangle h_n\| = 0$.) Show that there exists a linear isometry $f\colon H \to K$, so that

$$\langle f(g), f(h)\rangle = \langle g, h\rangle, \qquad \forall\, g, h \in H,$$

and such that f is injective (if $f(h) = 0$, then $h = 0$).

Solution. Let $\{h_n\}_{n\geq 0}$ be a countable orthonormal basis of H and let $\{k_n\}_{n\geq 0}$ be a countable orthonormal basis of K. Define a function f on this basis so that $f(h_n) = k_n$ for all $n \geq 0$. We can then extend f to all of H as follows. Any $h \in H$ can be written (uniquely) as $\sum_{n=0}^{\infty}\langle h, h_n\rangle h_n$ in the Hilbert space sense (i.e. $\lim_{m\to\infty} \|h - \sum_{n=0}^{m}\langle h, h_n\rangle h_n\| = 0$). So, define f on H by

$$f(h) = f\Big(\sum_{n=0}^{\infty}\langle h, h_n\rangle h_n\Big) := \sum_{n=0}^{\infty}\langle h, h_n\rangle f(h_n) = \sum_{n=0}^{\infty}\langle h, h_n\rangle k_n.$$

Since any h is uniquely written as $\sum_{n=0}^{\infty}\langle h, h_n\rangle h_n$, f is well-defined. Since the inner product function is linear, evidently the function f is linear (since it is the sum of linear functions). If $f(h) = 0$, then $\sum_{n=0}^{\infty}\langle h, h_n\rangle k_n = 0$. Since $\{k_n\}_{n\geq 0}$ is an orthonormal basis, the only way to express the 0 vector as a sum of this form is for all of the coefficients to be zero, i.e. $\langle h, h_n\rangle = 0$ for all $n \geq 0$. Consequently, $h = \sum_{n=0}^{\infty}\langle h, h_n\rangle h_n = 0$. Lastly, using continuity of the inner product

$$\begin{aligned}\langle f(g), f(h)\rangle &= \left\langle \sum_{n=0}^{\infty}\langle g, h_n\rangle k_n, \sum_{m=0}^{\infty}\langle h, h_m\rangle k_m \right\rangle = \sum_{n,m=0}^{\infty} \left\langle \langle g, h_n\rangle k_n, \langle h, h_m\rangle k_m \right\rangle \\ &= \sum_{n=0}^{\infty}\langle g, h_n\rangle\langle h, h_n\rangle\left\langle k_n, k_n \right\rangle = \sum_{n=0}^{\infty}\langle g, h_n\rangle\langle h, h_n\rangle\left\langle h_n, h_n \right\rangle \\ &= \sum_{n,m=0}^{\infty} \left\langle \langle g, h_n\rangle h_n, \langle h, h_m\rangle h_m \right\rangle = \langle g, h\rangle\end{aligned}$$