Graduate Statistics, 545, Spring 2020, USC Instructor: Steven Heilman

Name: ______________________ USC ID: ______________ Date: ___________

Signature: ______________________.
(By signing here, I certify that I have taken this test while refraining from cheating.)

Mid-Term 1

This exam contains 6 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may *not* use your book or any calculator on this exam. You *cannot* use your homeworks. You are required to show your work on each problem on the exam. The following rules apply:

- You have 50 minutes to complete the exam, starting at the beginning of class.
- **If you use a theorem or proposition from class or the notes or the book you must indicate this** and explain why the theorem may be applied. It is okay to just say, "by some theorem/proposition from class."
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

Do not write in the table to the right. Good luck![a]

1. (10 points) Find a sequence of functions $f_1, f_2, \ldots : [0,1] \to \mathbf{R}$ such that

$$\lim_{n\to\infty} \int_0^1 |f_n(x)|^2 \, dx = \infty,$$

and such that

$$\forall\, x \in [0,1], \quad \lim_{n\to\infty} f_n(x) = 0.$$

Prove your assertion. (In particular, you are asked to find a sequence of functions that converges pointwise to 0, such that this sequence does not converge to 0 in L_2.)

2. (10 points) Let X, Y be real-valued random variables with $\mathbf{E}\,|X| < \infty$. Let $f\colon \mathbf{R} \to \mathbf{R}$ be a bounded (measurable) function. Using the definition of conditional expectation, show that
$$\mathbf{E}(Xf(Y)\,|\,Y) = f(Y)\mathbf{E}(X|Y).$$

3. (10 points) Let $f(x) := x$ for all $x \in [-1/2, 1/2]$. Define $\widehat{f}(n) := \int_{-1/2}^{1/2} f(x) e^{-2\pi i n x} dx$ for all $n \in \mathbf{Z}$. Show that

$$\widehat{f}(n) = \frac{(-1)^{n+1}}{2\pi i n}, \qquad \forall\, n \in \mathbf{Z},\, n \neq 0.$$

Conclude that

$$\sum_{n \geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

(Hint: use Parseval's Identity.)

4. (10 points) Let H be a real n-dimensional inner product space over $\mathbf{R}$. Let K be a real n-dimensional inner product space over $\mathbf{R}$. As we discussed in class, H and K are linearly isometric, in the sense that: $\exists$ a linear function $f\colon H \to K$ such that

$$\langle f(g), f(h)\rangle = \langle g, h\rangle, \qquad \forall\, g, h \in H,$$

and such that f is injective (if $f(h) = 0$, then $h = 0$).

Extend this result to the infinite-dimensional setting. That is, let H, K be real, separable, infinite-dimensional Hilbert spaces over $\mathbf{R}$. (An infinite-dimensional Hilbert space H is separable if there exists a countable orthonormal basis $\{h_n\}_{n\geq 0} \subseteq H$, i.e. $\langle h_n, h_m\rangle = 1_{\{n=m\}}$ for all $n, m \geq 0$, and for any $h \in H$, $\lim_{m\to\infty} ||h - \sum_{n=0}^{m}\langle h, h_n\rangle h_n|| = 0$.) Show that there exists a linear isometry $f\colon H \to K$, so that

$$\langle f(g), f(h)\rangle = \langle g, h\rangle, \qquad \forall\, g, h \in H,$$

and such that f is injective (if $f(h) = 0$, then $h = 0$).

(Scratch paper)