Math 541b, Fall 2023, USC — Instructor: Steven Heilman

Name: ______________________ USC ID: ______________ Date: ___________

Signature: ______________________. Discussion Section: ______________________
(By signing here, I certify that I have taken this test while refraining from cheating.)

Exam 1

This exam contains 6 pages (including this cover page) and 4 problems. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- You have 50 minutes to complete the exam, starting at the beginning of class.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

Do not write in the table to the right. Good luck![a]

[a]September 18, 2023,

1. (10 points) Let X be a Gaussian random variable with mean $\mu \in \mathbf{R}$ and variance 1, so that X has PDF

$$f(x) := \frac{1}{\sqrt{2\pi}} e^{-(x-\mu)^2/2}, \qquad \forall\, x \in \mathbf{R}.$$

Provide a 99% confidence interval for μ.

(This confidence interval should be a function of X alone, i.e. it should only be a function of a single sample. Also, this confidence interval should be of the form $[X - c, X + c]$.)

Your answer should use the function $\Psi(t) := \int_{-t}^{t} e^{-x^2/2} dx/\sqrt{2\pi}$, $\Psi\colon (0,\infty) \to (0,1)$, and/or the function $\Psi^{-1}\colon (0,1) \to (0,\infty)$. (Recall that $\Psi(\Psi^{-1}(s)) = s$ for all $s \in (0,1)$ and $\Psi^{-1}(\Psi(t)) = t$ for all $t > 0$.)

2. (10 points) Suppose X is a binomial distributed random variable with parameters 2 and $\theta \in \{1/2, 3/4\}$. (That is, X is the number of heads that result from flipping two coins, where each coin has probability θ of landing heads.)
We want to test the hypothesis H_0 that $\theta = 1/2$ versus the hypothesis H_1 that $\theta = 3/4$.

Let $\mathcal{T}$ be the set of hypothesis tests with significance level at most 1/20.
(Recall that the significance level of a hypothesis test $\phi\colon \mathbf{R} \to [0,1]$ is $\sup_{\theta\in\Theta_0} \mathbf{E}_\theta \phi(X)$.)

Find a uniformly most powerful (UMP) class $\mathcal{T}$ hypothesis test $\phi\colon \mathbf{R} \to [0,1]$.
Compute all constants that appear in the definition of ϕ. Justify your answer.

Hint: you can freely use the following facts about the PMF f_θ of X

$$\frac{f_{3/4}(0)}{f_{1/2}(0)} = \frac{1}{4}, \qquad \frac{f_{3/4}(1)}{f_{1/2}(1)} = \frac{3}{4}, \qquad \frac{f_{3/4}(2)}{f_{1/2}(2)} = \frac{9}{4}.$$

3. (10 points) Let $X_1, \ldots, X_n$ be a real-valued random sample of size n so that X_1 has PDF given by

$$f(x) = \lambda e^{-\lambda x} 1_{x>0}, \qquad \forall\, x \in \mathbf{R},$$

where $\lambda > 0$ is an unknown parameter.

Suppose we want to test the hypothesis H_0 that $0 < \lambda \leq 2$ versus the hypothesis H_1 that $\lambda > 2$.

Describe the uniformly most powerful hypothesis test among all hypothesis tests with significance level at most $1/3$. Justify your answer.

(You do not have to calculate the exact constants that appear in the definition of the UMP test.)

(Hint: a monotone decreasing likelihood ratio for X is a monotone increasing likelihood ratio for $-X$.)

4. (10 points) Let $X_1, \ldots, X_n$ be a real-valued random sample of size n from a family of distributions $\{f_\theta : \theta \in \Theta\}$. (That is, X has distribution f_θ, where $X = (X_1, \ldots, X_n)$.) Suppose $\Theta = \mathbf{R}$. Fix $\theta \in \mathbf{R}$. Suppose $\{f_\theta : \theta \in \Theta\}$ has the monotone likelihood ratio property with respect to a statistic $Y = t(X)$ that is a continuous random variable.

   Consider the set of hypothesis tests with rejection region $\{x \in \mathbf{R}^n : t(x) > c\}$, where $c \in \mathbf{R}$ is a constant (so that different values of c correspond to different hypothesis tests.) Fix $\theta_0 \in \Theta$. Suppose we are testing $H_0 = \{\theta \leq \theta_0\}$ versus $H_1 = \{\theta > \theta_0\}$. For any $0 < \alpha < 1$, let $c_\alpha \in \mathbf{R}$ such that the rejection region $\{x \in \mathbf{R}^n : t(x) > c_\alpha\}$ has significance level α. Define the p-value quantity

$$p(x) := \inf\{\alpha \in [0,1] : t(x) > c_\alpha\}, \qquad \forall\, x \in \mathbf{R}^n.$$

   (Here we define the infimum of the empty set to be 1.)

   Show that, if $X = x$, then $p(x)$ satisfies

$$p(x) = \mathbf{P}_{\theta_0}(t(X) > t(x)).$$

(Scratch paper)