Graduate Statistics, 541A, Fall 2022, USC — Instructor: Steven Heilman

Name: ____________ USC ID: __________ Date: ________

Signature: ____________.
(By signing here, I certify that I have taken this test while refraining from cheating.)

Final Exam

This exam contains 13 pages (including this cover page) and 8 problems. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- You have 120 minutes to complete the exam, starting at the beginning of class.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Do not write in the table to the right. Good luck![a]

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total:	80	

Reference sheet

Below are some definitions that may be relevant.

We say that a sequence of random variables $X_1, X_2, \ldots : \Omega \to \mathbf{R}$ **converges in probability** to a random variable $X : \Omega \to \mathbf{R}$ if: for all $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbf{P}(|X_n - X| > \varepsilon) = 0.$$

We say that a sequence of real-valued random variables $X_1, X_2, \ldots$ **converges in distribution** to a real-valued random variable X if, for any $t \in \mathbf{R}$ such that $\mathbf{P}(X \leq t)$ is continuous at t,

$$\lim_{n\to\infty} \mathbf{P}(X_n \leq t) = \mathbf{P}(X \leq t).$$

Suppose $X = (X_1, \ldots, X_n)$ is a random sample of size n from a distribution f where $f \in \{f_\theta : \theta \in \Theta\}$ is a family of densities (such as an exponential family). Let $t : \mathbf{R}^n \to \mathbf{R}^k$, so that $Y := t(X_1, \ldots, X_n)$ is a statistic.

We say that Y is a **sufficient statistic** for θ if, for every $y \in \mathbf{R}^k$ and for every $\theta \in \Theta$, the conditional distribution of $(X_1, \ldots, X_n)$ given $Y = y$ (with respect to probabilities given by f_θ) does not depend on θ.

We say Y is **minimal sufficient** for θ if Y is sufficient for θ and, for every statistic $Z : \Omega \to \mathbf{R}^m$ that is sufficient for θ, there exists a function $r : \mathbf{R}^m \to \mathbf{R}^k$ such that $Y = r(Z)$.

We say Y is **complete** for $\{f_\theta : \theta \in \Theta\}$ if the following holds:

For any $f : \mathbf{R}^m \to \mathbf{R}$ such that $\mathbf{E}_\theta f(Y) = 0 \quad \forall \, \theta \in \Theta$, it holds that $f(Y) = 0$.

We say Y is **ancillary** for θ if the distribution of Y does not depend on θ.

Let $X, Y, Z : \Omega \to \mathbf{R}$ be discrete or continuous random variables. Let A be the range of Y. Define $g : A \to \mathbf{R}$ by $g(y) := \mathbf{E}(X|Y = y)$, for any $y \in A$. We then define the **conditional expectation** of X given Y, denoted $\mathbf{E}(X|Y)$, to be the random variable $g(Y)$.

Let $\{f_\theta : \theta \in \Theta\}$ be a family of multivariable probability densities or probability mass functions. Assume $\Theta \subseteq \mathbf{R}$. Let X be a random vector with distribution f_θ. Define the **Fisher information** of the family to be

$$I(\theta) = I_X(\theta) := \mathbf{E}_\theta\left(\frac{d}{d\theta} \log f_\theta(X)\right)^2, \qquad \forall \, \theta \in \Theta,$$

if this quantity exists and is finite.

1. (10 points) Let $X_1, X_2, \ldots$ be real-valued random variables that converge in probability to a constant $a \in \mathbf{R}$.

   Let $h\colon \mathbf{R} \to \mathbf{R}$ be a continuous function. (For any $x \in \mathbf{R}$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, if $y \in \mathbf{R}$ satisfies $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.)

   Show that $h(X_1), h(X_2), \ldots$ converges in probability to $h(a)$.

   [This was a repeated and simplified homework question.]

2. (10 points) Let $X_1, \ldots, X_n$ be i.i.d. random variables, so that X_1 has PDF $f_\theta \colon \mathbf{R} \to [0, \infty)$, where $\theta = (\theta_1, \theta_2) \in \mathbf{R}^2$ is an unknown parameter.

   Let Y be a statistic (so that Y is a function of $X_1, \ldots, X_n$). In all cases below, as usual, you must **justify your answer**.

   (i) Suppose Y is sufficient for θ. Is it true that Y is sufficient for θ_1?

   (ii) Suppose Y is sufficient for θ_1, and Y is sufficient for θ_2. Is it true that Y is sufficient for θ?

   (iii) Suppose Y is minimal sufficient for θ_1, and Y is minimal sufficient for θ_2. Is it true that Y is minimal sufficient for θ?

3. (10 points) Find real-valued random variables X, Y, $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ such that the following holds.

   - $X_1, X_2, \ldots$ converges in distribution to X.
   - $Y_1, Y_2, \ldots$ converges in distribution to Y.
   - $X_1 + Y_1, X_2 + Y_2, \ldots$ does **NOT** converges in distribution to any random variable.

   Prove your assertions.

   [This question was discussed in the final exam review session.]

4. (10 points) Let $X_1, \ldots, X_n$ be i.i.d. random variables, so that X_1 has PDF $f_\theta \colon \mathbf{R} \to [0, \infty)$, where $\theta > 0$ is an unknown parameter and

$$f_\theta(x) := \frac{\theta^2}{2} e^{-\theta^2 |x|}, \qquad \forall\, x \in \mathbf{R}.$$

- Find the MLE Y_n of θ. [Warning: do NOT find the MLE of θ^2.]
- Compute the Fisher information $I_{X_1}(\theta)$.
  (You can freely use without proof that $\mathbf{E}X_1^2 = 2\theta^{-4}$ and $\mathbf{E}\,|X_1| = \theta^{-2}$.)
- Find a random variable Z such that $\sqrt{n}(Y_n - \theta)$ converges in distribution to Z as $n \to \infty$. (You can freely use without proof that $Y_1, Y_2, \ldots$ converges in probability to θ.)

[This was a modified qual exam question.]

5. (10 points) Let $X_1, \ldots, X_n$ be i.i.d. random variables, so that X_1 has PDF $f_\theta\colon \mathbf{R} \to [0,\infty)$, where $\theta > 0$ is an unknown parameter and

$$f_\theta(x) := \frac{2x}{\theta^2}, \qquad \forall\, 0 \leq x \leq \theta.$$

- Find a method of moments estimator Y_n of θ. Is Y_n unbiased?
- Find constants a_n, b_n such that $a_n(Y_n - b_n)$ converges in distribution as $n \to \infty$ to a mean zero variance one Gaussian random variable.
  (Hint: you should find that $\mathrm{Var} X_1 = \theta^2/18$.)

[This was a shortened qual exam question.]

6. (10 points) Let $X_1, \ldots, X_n$ be i.i.d. Gaussian random variables with unknown mean $\mu \in \mathbf{R}$ and (known) variance 1.

In this problem, you can freely use that the sample mean $M_n := \frac{1}{n}\sum_{i=1}^n X_i$ is complete and sufficient for μ.

- Find a minimal sufficient statistic for μ.
- Find an unbiased estimator Y_n of the quantity $\mathbf{P}_\mu(X_1 \leq 0)$.
- Using any method you want to use, find the UMVU of the quantity $\mathbf{P}_\mu(X_1 \leq 0)$.

[This was a shortened qual exam question.]

7. (10 points) Let $X_1, \ldots, X_n$ be i.i.d. random variables that are uniformly distributed in $[\theta - 1/2, \theta + 1/2]$, where $\theta \in \mathbf{R}$ is unknown. Note that $\mathbf{E}X_1 = \theta$, $\mathbf{E}X_1^2 = \theta^2 + 1/12$.

   - Give two different method of moments estimators that estimate θ.
   - Show that an MLE for θ is not unique. That is, describe two different maximum likelihood estimators for θ.
   - Is the Fisher information $I_{X_1}(\theta)$ well-defined? Explain.
   - Show that any MLE for θ is consistent.

   [This was a modified homework question.]

8. (10 points) Prove the Cramér-Rao inequality:

   Let $X\colon \Omega \to \mathbf{R}^n$ be a random variable with distribution from a family of multivariable PDFs $\{f_\theta\colon \theta \in \Theta\}$ with $\Theta \subseteq \mathbf{R}$. Let $t\colon \mathbf{R}^n \to \mathbf{R}$ and let $Y := t(X)$ be statistic. For any $\theta \in \Theta$ let $g(\theta) := \mathbf{E}_\theta Y$. Then

$$\mathrm{Var}_\theta(Y) \geq \frac{|g'(\theta)|^2}{I_X(\theta)}, \qquad \forall\, \theta \in \Theta.$$

   Moreover, if $I_X(\theta) = 0$, then $g'(\theta) = 0$.

   (You are allowed to differentiate under any integral in your proof. Also, we assume that $\{x \in \mathbf{R}^n\colon f_\theta(x) > 0\}$ does not depend on θ, and for a.e. $x \in \mathbf{R}^n$, $(d/d\theta) f_\theta(x))$ exists and is finite, and the Fisher information satisfies any identity we have ever shown it to satisfy in this course.)

(Scratch paper 1)

(Scratch paper 2)

(Scratch paper 3)