507A Midterm 1 Solutions[1]

1. Question 1

Let $X_1, X_2, \ldots \Omega \to \mathbf{R}$ be random variables such that $\mathbf{E}X_i = 0$ and $\mathbf{E}X_i^2 = 1$ for all $i \geq 1$. Show that

$$\mathbf{P}(X_n > n \text{ for infinitely many } n \geq 1) = 0.$$

Solution. Note that $\mathrm{var}(X_n) = \mathbf{E}X_n^2 - (\mathbf{E}X_n)^2 = 1$ for all $n \geq 1$. From Chebyshev's inequality,

$$\mathbf{P}(X_n > n) \leq \frac{\mathrm{var}(X_n)}{n^2} = \frac{1}{n^2}, \qquad \forall\, n \geq 1.$$

Therefore,

$$\sum_{n\geq 1} \mathbf{P}(X_n > n) \leq \sum_{n \geq 1} \frac{1}{n^2} < \infty.$$

So, from the Borel-Cantelli Lemma, the claim follows.

2. Question 2

In this Exercise we will use the following form of Jensen's inequality (which you can take as a given fact):

Let $X\colon \Omega \to [-\infty, \infty]$ be a random variable. Let $\phi\colon \mathbf{R} \to \mathbf{R}$ be convex. Assume that $\mathbf{E}\,|X| < \infty$ and $\mathbf{E}\,|\phi(X)| < \infty$. Then

$$\phi(\mathbf{E}X) \leq \mathbf{E}\phi(X).$$

In this Exercise, the above form of Jensen's inequality is the **only** form of Jensen's inequality that you are allowed to use.

Prove: if $\mathbf{E}(X^2) < \infty$, then $|\mathbf{E}X| < \infty$.

Solution. Fix $n \geq 1$ and let $X_n := \max(-n, \min(X, n))$. That is, $X_n = X$ when $|X| < n$, $X_n = n$ when $X \geq n$ and $X_n = -n$ when $X \leq -n$. Then $|X_n|$ increases monotonically to $|X|$ as $n \to \infty$. Since X_n is bounded, $\mathbf{E}\,|X_n| < \infty$ and $\mathbf{E}X_n^2 < \infty$, so Jensen's inequality implies that

$$(\mathbf{E}X_n)^2 \leq \mathbf{E}X_n^2.$$

By the Monotone Convergence Theorem, we can let $n \to \infty$ on the right side to get

$$(\mathbf{E}X_n)^2 \leq \mathbf{E}X^2 < \infty, \qquad \forall\, n \geq 1. \qquad (*)$$

Moreover, $\mathbf{E}X_n^2 \leq \mathbf{E}X^2 < \infty$, for all $n \geq 1$. The Convergence Theorem with Bounded Moment (Theorem 1.59 in the notes) then implies that $\mathbf{E}X = \mathbf{E}\lim_{n\to\infty} X_n = \lim_{n\to\infty} \mathbf{E}X_n$. That is, we can let $n \to \infty$ on the left side of $(*)$ to get

$$(\mathbf{E}X)^2 \leq \mathbf{E}X^2.$$

[1]September 21, 2020,

3. Question 3

Let Ω be a universal set, and let $\mathcal{F}$ be a set of subsets of Ω. Suppose $\mathcal{F}$ is a monotone class.

(As usual, we define $\sigma(\mathcal{F})$ to be the σ-algebra generated by $\mathcal{F}$. That is, $\sigma(\mathcal{F})$ is the smallest σ-algebra containing $\mathcal{F}$.)

Prove or disprove the following statement:

$\sigma(\mathcal{F})$ is the smallest monotone class containing $\mathcal{F}$.

Solution. This statement is false. Let $\mathcal{F}$ be the set of all two-element subsets of $\{1,2,3\}$. Then $\mathcal{F}$ is trivially a monotone class. (For example, the only increasing unions of sets in $\mathbb{F}$ consist of singleton sets, e.g. $\cup_{n=1}^{\infty}\{1,2\} = \{1,2\}$.) However, $\sigma(F)$ is strictly larger than $\mathcal{F}$, since $\sigma(F)$ consists of all subsets of $\{1,2,3\}$.

4. Question 4

Give an example of a function $f\colon [-1,1]\times[-1,1] \to \mathbf{R}$ such that

- For a.e. $x \in [-1,1]$, $\int_{-1}^{1} f(x,y)dy = 0$.
- For a.e. $y \in [-1,1]$, $\int_{-1}^{1} f(x,y)dx = 0$.
- $\int_{[-1,1]\times[-1,1]} |f(x,y)|\, dxdy = \infty$.

That is, $\int_{-1}^{1}\left(\int_{-1}^{1} f(x,y)dy\right)dx = 0$, $\int_{-1}^{1}\left(\int_{-1}^{1} f(x,y)dx\right)dy = 0$, while

$$\int_{[-1,1]\times[-1,1]} |f(x,y)|\, dxdy = \infty.$$

Consequently, a "converse" of Fubini's Theorem does not hold for probability spaces. (We can divide by appropriate constants to turn these integrals into expected values.)

Solution. Here is one example that works. Let $r(x,y)$ be the radial function such that $r(x,y) = 1$ in the first and third quadrants ($x > 0, y > 0$ and $x < 0, y < 0$) and $r(x,y) = -1$ in the second and fourth quadrants ($x > 0, y < 0$ and $x > 0, y < 0$). (And set $r(x,y) = 0$ when $x = 0$ or $y = 0$.) Let $g(x,y) := (x^2+y^2)^{-1}$ for any $x, y \neq 0$ (and let $g(0,0) := 0$). Define

$$f(x,y) := r(x,y)g(x,y), \qquad \forall -1 \leq x, y \leq 1.$$

Since $g(x,y) = g(-x,y) = g(x,-y)$ while $r(x,y) = -r(x,-y) = -r(-x,y)$ for all $-1 \leq x, y \leq 1$, the function f then satisfies $f(x,y) = -f(x,-y) = -f(-x,y)$ for all $-1 \leq x, y \leq 1$, so the first two properties follow immediately. For the third property, note that

$$\int_{[-1,1]\times[-1,1]} |f(x,y)|\, dxdy = \int_{[-1,1]\times[-1,1]} |g(x,y)|\, dxdy \geq \int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^{-2} r\,dr\,d\theta = \infty.$$