Please provide complete and well-written solutions to the following exercises.

Due March 26, 12PM noon PST, to be uploaded as a single PDF document to blackboard (under the Assignments tab).

Homework 5

Exercise 1. Using the Optional Stopping Theorem, prove Wald's equations:

Let $X_1, X_2, \ldots : \mathcal{C} \to \mathbf{R}$ be i.i.d. Let N be a stopping time. Let $S_0, S_1, \ldots$ be the corresponding random walk with $S_0 := 0$.

- If $\mathbf{E}N < \infty$, and $\mathbf{E}\,|X_1| < \infty$, then $\mathbf{E}S_N = \mathbf{E}X_1\mathbf{E}N$.
- If $\mathbf{E}X_1 = 0, \mathbf{E}X_1^2 < \infty$ and $\mathbf{E}N < \infty$, then $\mathbf{E}S_N^2 = \mathbf{E}X_1^2\mathbf{E}N$.

Exercise 2. Let $1/2 < p < 1$. Consider the random walk on $\mathbf{Z}$ such that $\mathbf{P}(X_1 = 1) = p$ and $\mathbf{P}(X_1 = -1) = 1 - p$. Let $S_0, S_1, \ldots$ be the corresponding random walk with $S_0 := 0$. Let $N := \min\{n \geq 1 \colon S_n > 0\}$. Using Wald's equation for $\min(N, n)$ and then letting $n \to \infty$, show that $\mathbf{E}N = 1/\mathbf{E}X_1 = 1/(2p - 1)$.

Exercise 3. Let $X_0 = 0$. Let $(X_0, X_1, \ldots)$ such that $\mathbf{P}(X_i = 1) = \mathbf{P}(X_i = -1) = 1/2$ for all $i \geq 1$. For any $n \geq 0$, let $Y_n = X_0 + \cdots + X_n$. So, $(Y_0, Y_1, \ldots)$ is a symmetric simple random walk on $\mathbf{Z}$. Show that $Y_n^2 - n$ is a martingale (with respect to $(X_0, X_1, \ldots)$).

Exercise 4. Let $1/2 < p < 1$. Let $(X_0, X_1, \ldots)$ such that $\mathbf{P}(X_i = 1) = p$ and $\mathbf{P}(X_i = -1) = 1 - p$ for all $i \geq 1$. For any $n \geq 0$, let $Y_n = X_0 + \cdots + X_n$. Let $T_0 = \min\{n \geq 1 \colon Y_n = 0\}$. Prove that $\mathbf{P}_1(T_0 = \infty) > 0$. Then, deduce that $\mathbf{P}_0(T_0 = \infty) > 0$. That is, there is a positive probability that the biased random walk never returns to 0, even though it started at 0.

Exercise 5 (Ballot Theorem)**.** Let a, b be positive integers. Suppose there are c votes cast by c people in an election. Candidate 1 gets a votes and candidate 2 gets b votes. (So $c = a + b$.) Assume $a > b$. The votes are counted one by one. The votes are counted in a uniformly random ordering, and we would like to keep a running tally of who is currently winning. (News agencies seem to enjoy reporting about this number.) Suppose the first candidate eventually wins the election. We ask: with what probability will candidate 1 always be ahead in the running tally of who is currently winning the election? As we will see, the answer is $\frac{a-b}{a+b}$.

To prove this, for any positive integer k, let S_k be the number of votes for candidate 1, minus the number of votes for candidate 2, after k votes have been counted. Then, define $X_k := S_{c-k}/(c - k)$. Show that $X_0, X_1, \ldots$ is a martingale with respect to $S_c, S_{c-1}, S_{c-2}, \ldots$. Then, let T such that $T = \min\{0 \leq k \leq c \colon X_k = 0\}$, or $T = c - 1$ if no such k exists. Apply the Optional Stopping theorem to X_T to deduce the result.

Exercise 6. Let $X_1, X_2, \ldots$ be i.i.d. random variables with $\mathbf{E}X_i = 0$ for every $i \geq 1$. Suppose there exists $\sigma > 0$ such that $\mathrm{Var}(X_i) = \sigma^2$ for all $i \geq 1$. For any $n \geq 1$, let $S_n = X_1 + \cdots + X_n$. Show that $S_n^2 - n\sigma^2$ is a martingale with respect to $X_1, X_2, \ldots$. (We let $X_0 = 0$.)

Let $a > 0$. Let $T = \min\{n \geq 1 \colon |S_n| \geq a\}$. Using the Optional Stopping Theorem, show that $\mathbf{E}T \geq a^2/\sigma^2$. Observe that a simple random walk on $\mathbf{Z}$ has $\sigma^2 = 1$ and $\mathbf{E}T = a^2$ when $a \in \mathbf{Z}$.

Exercise 7 (Azuma's Inequality)**.** In this exercise, we prove a generalization of the Hoeffding inequality to martingales. Let $c_1, c_2, \ldots > 0$. Let $(X_0, X_1, \ldots)$ be a martingale. Assume that $|X_n - X_{n-1}| \leq c_n$ for all $n \geq 1$. Then for any $t > 0$,

$$\mathbf{P}(|X_n - X_0| > t) \leq 2e^{-\frac{t^2}{2\sum_{i=1}^n c_i^2}}.$$

Prove this inequality using the following steps.

- Let $\alpha > 0$. Show that $\mathbf{E}e^{\alpha(X_n - X_0)} = \mathbf{E}[e^{\alpha(X_{n-1} - X_0)}\mathbf{E}(e^{\alpha(X_n - X_{n-1})}|\mathcal{F}_{n-1})]$. (When Y is a random variable, we denote $\mathbf{E}(Y|\mathcal{F}_n) := g(X_0, \ldots, X_n)$ where $g(x_0, \ldots, x_n) := \mathbf{E}(Y|X_0 = x_0, \ldots, X_n = x_n)$ for any $x_0, \ldots, x_n \in \mathbf{R}$.)
- For any $y \in [-1, 1]$, show that $e^{\alpha c_n y} \leq \frac{1+y}{2}e^{\alpha c_n} + \frac{1-y}{2}e^{-\alpha c_n}$.
- Take the conditional expectation of this inequality when $y = (X_n - X_{n-1})/c_n$.
- Now argue as in Hoeffding's inequality.

Using Azuma's inequality, deduce **McDiarmid's Inequality**. Let $X_1, \ldots, X_n$ be independent real-valued random variables. Let $c_1, c_2, \ldots > 0$. Let $f\colon \mathbf{R}^n \to \mathbf{R}$ be a measurable function such that, for any $1 \leq m \leq n$,

$$\sup_{x_1, \ldots, x_{m-1}, x_m, x_m', x_{m+1}, \ldots, x_n \in \mathbf{R}} |f(x_1, \ldots, x_n) - f(x_1, \ldots, x_{m-1}, x_m', x_{m+1}, \ldots, x_n)| \leq c_m.$$

Then, for any $t > 0$,

$$\mathbf{P}(|f(X_1, \ldots, X_n) - \mathbf{E}f(X_1, \ldots, X_n)| > t) \leq 2e^{-\frac{t^2}{2\sum_{i=1}^n c_i^2}}.$$

(Note that a linear function f recovers Hoeffding's inequality, Theorem **??**.)