Please provide complete and well-written solutions to the following exercises.

Due March 12, 12PM noon PST, to be uploaded as a single PDF document to blackboard (under the Assignments tab).

Homework 4

Exercise 1. Let μ,ν be probability distributions on a finite state space Ω. Show that

$$||\mu-\nu||_{\mathrm{TV}}=\frac{1}{2}\sum_{x\in\Omega}|\mu(x)-\nu(x)|\,.$$

(Hint: consider the set $A=\{x\in\Omega\colon \mu(x)\geq\nu(x)\}$.)

Exercise 2. Let $(X_0,X_1,\ldots)$ be the simple random walk on $\mathbf{Z}$. Show that $\mathbf{P}_0(X_n=0)$ decays like $1/\sqrt{n}$ as $n\to\infty$. That is, show

$$\lim_{n\to\infty}\sqrt{2n}\,\mathbf{P}_0(X_{2n}=0)=\sqrt{\frac{2}{\pi}}.$$

Also, show the upper bound

$$\mathbf{P}_0(X_n=k)\leq\frac{10}{\sqrt{n}},\qquad\forall\, n\geq 0,\, k\in\mathbf{Z}.$$

(Hint 1: first consider the case $n=2r$ for $r\in\mathbf{Z}$. It may be helpful to show that $\binom{2r}{r+j}$ is maximized when $j=0$. To eventually deal with k odd, just condition on the first step of the walk.)

(Hint 2: you can freely use **Stirling's formula**:

$$\lim_{n\to\infty}\frac{n!}{\sqrt{2\pi n}(n/e)^n}=1.$$

Or, there is a more precise estimate: for any $n\geq 3$, there exists $1/(12n+1)\leq\varepsilon_n\leq 1/(12n)$ such that

$$n!=\sqrt{2\pi}e^{-n}n^{n+1/2}e^{\varepsilon_n}.)$$

Exercise 3. Show that every state in the simple random walk on $\mathbf{Z}$ is recurrent. (You should show this statement for any starting location of the Markov chain, i.e. show that $\mathbf{P}_x(T_y<\infty)=1$ for all $x,y\in\mathbf{Z}$.)

Then, find a nearest-neighbor random walk on $\mathbf{Z}$ such that every state is transient.

Exercise 4. For the simple random walk on $\mathbf{Z}$, show that $\mathbf{E}_0T_0=\infty$. Conclude that, for any $x,y\in\mathbf{Z}$, $\mathbf{E}_xT_y=\infty$.

Exercise 5. Let $(X_0, X_1, \ldots)$ be the "corner walk" on $\mathbf{Z}^2$. The transitions are described as follows. From any point $(x, y) \in \mathbf{Z}^2$, the Markov chain adds any of the following four vector to (x, y) each with probability $1/4$: $\{(1, 1,), (1, -1), (-1, 1), (-1, -1)\}$. Using that the coordinates of this walk are each independent simple random walks on $\mathbf{Z}$, conclude that there exists $c > 0$ such that

$$\lim_{n\to\infty} n\mathbf{P}_{(0,0)}(X_{2n} = (0, 0)) = c.$$

That is, $\mathbf{P}_{(0,0)}(X_{2n} = (0, 0))$ is about c/n, when n is large.

Now, note that the usual nearest-neighbor simple random walk on $\mathbf{Z}^2$ is a rotation of the corner walk by an angle of $\pi/4$. So, the above limiting statement also holds for the simple random walk on $\mathbf{Z}^2$.

Exercise 6. Let $S_0, S_1, \ldots$ be a random walk with $S_0 = 0$. Let Y be the number of times the random walk takes the value 0. Let $T_0 := \min\{n \geq 1 \colon S_n = 0\}$.

- Y is a geometric random variable with success probability $\mathbf{P}(T_0 = \infty)$.
- $\mathbf{E}Y = \frac{1}{\mathbf{P}(T_0=\infty)}$. (Here we interpret $1/0$ as ∞.)

(Hint: $\{Y = k\} = \{T_0^{(k-1)} < \infty, T_0^{(k)} = \infty\} = \{T_0^{(k-1)} < \infty, T_0^{(k)} - T_0^{(k-1)} = \infty\}$.)

Exercise 7. Let $(X_0, X_1, \ldots)$ be a finite, irreducible Markov chain with transition matrix P and state space Ω. For any $x, y \in \Omega$, define

$$G(x, y) := \mathbf{E}_x \sum_{n=0}^{\infty} 1_{\{X_n=y\}} = \sum_{n=0}^{\infty} \mathbf{P}^n(x, y)$$

to be the expected number of visits to y starting from x. Show that the following are equivalent:

(i) $G(x, x) = \infty$ for some $x \in \Omega$.
(ii) $G(x, y) = \infty$ for all $x, y \in \Omega$.
(iii) $\mathbf{P}_x(T_x < \infty)$ for some $x \in \Omega$.
(iii) $\mathbf{P}_x(T_y < \infty)$ for all $x, y \in \Omega$.

So, in an irreducible finite Markov chain, a single state is recurrent if and only if all states are recurrent.

Exercise 8. Show that if the Simple Random Walk on $\mathbf{Z}^d$ is recurrent, then this random walk takes every value in $\mathbf{Z}^d$ infinitely many times (with probability 1). And if the Simple Random Walk on $\mathbf{Z}^d$ is transient, then this random walk takes any fixed value in $\mathbf{Z}^d$ only finitely many times (with probability 1).

Exercise 9. Let $0 < p < 1$. Consider the random walk on $\mathbf{Z}$ such that $\mathbf{P}(X_1 = 1) = p$ and $\mathbf{P}(X_1 = -1) = 1-p$. Show that the corresponding random walk $S_0, S_1, \ldots$ is transient when $p \neq 1/2$.

Exercise 10. Let $S_0, S_1, \ldots$ and $S_0', S_1', \ldots$ be independent simple random walks on $\mathbf{Z}^d$. Let $N := \sum_{n,m\geq 0} 1_{S_n=S_m'}$ be the number of pairs of intersections of these two random walks. For any $y \in \mathbf{R}^d$, let $\phi(y) := \mathbf{E}e^{i\langle y, X_1\rangle}$.

- Show $\mathbf{E}N = \lim_{s\to 1^-} \int_{[-\pi,\pi]^d} \frac{1}{|1-s\phi(y)|^2} \frac{dy}{(2\pi)^d}$. (Hint: consider $\mathbf{E}e^{i\langle y,(S_n - S'_m)\rangle}$.)
- For what $d \geq 1$ is $\mathbf{E}N < \infty$?
- Let $C := \{S_n : n \geq 0\} \cap \{S'_n : n \geq 0\}$ be the intersection set of the two independent random walks. Let $|C|$ denote the cardinality of C. Show that if the simple random walk on $\mathbf{Z}^d$ is transient, then $\mathbf{P}(N = \infty) = 1$ if and only if $\mathbf{P}(|C| = \infty) = 1$. (Hint: $N = \sum_{x\in C} N_x N'_x$ where $N_x := \sum_{n\geq 0} 1_{S_n = x}$ is the number of visits of the first random walk to x.) In the recurrent case $d = 1, 2$, Exercise 8 implies that $\mathbf{P}(|C| = \infty) = 1$. For any $d \geq 1$, note that $N < \infty$ implies $|C| < \infty$. It can also be shown that $\mathbf{P}(N < \infty) \in \{0, 1\}$, $\mathbf{P}(|C| = \infty) \in \{0, 1\}$, and that $\mathbf{P}(N < \infty) = 1$ if and only if $\mathbf{E}N < \infty$ (you don't have to show these things). In summary, $\mathbf{P}(|C| = \infty) = 1$ if and only if $\mathbf{E}N = \infty$.
- Hypothesize what happens to $\mathbf{E}N$ when we instead consider the tuples of intersections of $k > 2$ independent simple random walks in $\mathbf{R}^d$. (You don't have to prove your hypothesis.)