Graduate Applied Probability II, Spring 2021, USC Instructor: Steven Heilman

Name: ______________________ USC ID: ______________ Date: ___________

Signature: ______________________.
(By signing here, I certify that I have taken this test while refraining from cheating.)

Exam 1

This exam contains 7 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may use your books and notes on this exam. You cannot use a calculator or any other electronic device (or internet-enabled device) on this exam. You are required to show your work on each problem on the exam. The following rules apply:

- You have 24 hours to complete the exam.
- **If you use a theorem or proposition from class or the notes or the book you must indicate this** and explain why the theorem may be applied. It is okay to just say, "by some theorem/proposition from class."
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper is at the end of the exam.

Do not write in the table to the right. Good luck![a]

Problem	Points	Score
1	10	
2	10	
3	20	
4	20	
Total:	60	

1. (10 points) Consider a standard 8×8 chess board. Let V be a set of vertices corresponding to each square on the board (so V has 64 elements). Any two vertices $x, y \in V$ are connected by an edge if and only if a king can move from x to y. (The king chess piece can move either one space along the horizontal, one space along the vertical axis, or one space diagonally.) Consider the simple random walk on this graph. This Markov chain then represents a king randomly moving around a chess board. For every space x on the chessboard, compute the expected return time $\mathbf{E}_x T_x$ for that space. (It might be convenient to just draw the expected values on the chessboard itself.)

2. (10 points) Let Ω be a finite state space. This problem demonstrates that the total variation distance is a metric. That is, show that the following three properties are satisfied:

   - $||\mu - \nu||_{\mathrm{TV}} \geq 0$ for all probability distributions μ, ν on Ω, and $||\mu - \nu||_{\mathrm{TV}} = 0$ if and only if $\mu = \nu$.
   - $||\mu - \nu||_{\mathrm{TV}} = ||\nu - \mu||_{\mathrm{TV}}$
   - $||\mu - \nu||_{\mathrm{TV}} \leq ||\mu - \eta||_{\mathrm{TV}} + ||\eta - \nu||_{\mathrm{TV}}$ for all probability distributions μ, ν, η on Ω.

   (Hint: you may want to use the triangle inequality for real numbers: $|x - y| \leq |x - z| + |z - y|, \forall\, x, y, z \in \mathbf{R}$.)

3. (20 points) Suppose we have a Markov Chain $(X_0, X_1, \ldots)$ with state space $\Omega = \{1, 2, 3\}$ and with the following transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

- Is the Markov chain irreducible? Prove your assertion.
- Classify all states in the Markov chain as recurrent or transient.
- Is the Markov chain aperiodic? Prove your assertion.
- List all eigenvalues of the matrix P.
- Find a stationary distribution π, if it exists. (If it does not exist, prove it.) Is the stationary distribution unique?

4. (20 points) (In this problem, you are allowed to use a computer or calculator for the purpose of computing powers of the matrix P.)

Suppose we have a Markov Chain $(X_0, X_1, \ldots)$ with state space $\Omega = \{1, 2, 3, 4, 5, 6\}$ and with the following transition matrix

$$P = \begin{pmatrix} 0 & 0 & .1 & .2 & .4 & .3 \\ 0 & 0 & .1 & .2 & .4 & .3 \\ .3 & .4 & .3 & 0 & 0 & 0 \\ .1 & .2 & .4 & .3 & 0 & 0 \\ 0 & .1 & .2 & .4 & .3 & 0 \\ 0 & 0 & .1 & .2 & .4 & .3 \end{pmatrix}.$$

- Is the Markov chain irreducible? Prove your assertion.
- Classify all states in the Markov chain as recurrent or transient.
- Is the Markov chain aperiodic? Prove your assertion.
- Find a stationary distribution π, if it exists. (If it does not exist, prove it.) (If it does exist, you can just report each entry of π to four decimal places of accuracy.) Is the stationary distribution unique?

(Scratch paper)

(Extra Scratch paper)