MATH 499, GAME THEORY, HOMEWORK SOLUTIONS

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1. Homework 1

Exercise 1.4. Prove the following assertion by induction:

For any natural number n, $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

Solution. For the base case, $n = 1$, the right side of the equation above gives

$$\frac{1}{6}(1)(1+1)(2\cdot 1+1) = \frac{1}{6}(1)(2)(3) = 1,$$

so the equation holds. For the inductive step, suppose the equation holds for $n-1$. That is

$$(*) \quad 1^2 + 2^2 + \cdots + (n-1)^2 = \frac{1}{6}(n-1)(n-1+1)(2(n-1)+1) = \frac{1}{6}(n-1)n(2n-1). \quad (1)$$

By the inductive hypothesis (*), we can write

$$\begin{aligned}1^2 + 2^2 + \cdots + (n-1)^2 + n^2 &= \frac{1}{6}(n-1)n(2n-1) + n^2\\ &= \frac{1}{6}(2n^3 - 3n^2 + n) + n^2\\ &= \frac{1}{6}(2n^3 - 3n^2 + n + 6n^2)\\ &= \frac{1}{6}(2n^3 + 3n^2 + n)\\ &= \frac{1}{6}n(n+1)(2n+1).\end{aligned}$$

Thus, by induction, the desired equation holds for all n. □

Exercise 1.5. Write the following numbers in binary: 1,3,5,8,9. Compute the following sums modulo 2: $1 + 3$, $4 + 9$, $10^{10} + 1$.

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Solution. One can verify the following equations:

$$1_{10} = 1_2$$
$$3_{10} = 11_2$$
$$5_{10} = 101_2$$
$$8_{10} = 1000_2$$
$$9_{10} = 1001_2$$

For example,

$$5_{10} = 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 101_2.$$

Modulo 2, we compute

$$1 + 3 = 1 + 1 = 2 = 0 \pmod 2$$
$$4 + 9 = 0 + 1 = 1 \pmod 2$$
$$10^{10} + 1 = 0^{10} + 1 = 0 + 1 = 1 \pmod 2.$$

□

Exercise 1.6. Tic-tac-toe is a partisan combinatorial game. Prove that it is progressively bounded. Then, try to figure out which of the following situations is true: the first player has a winning strategy; the second player has a winning strategy; both players have a strategy forcing at least a draw. (You do not need to give a formal proof of which situation holds, just try to guess the answer by playing a few tic-tac-toe games and by drawing on your personal experience.)

Solution. To prove that Tic-tac-toe is progressively bounded, we must argue that for any starting position x, the game must terminate after some finite number $B(x)$ steps (see definition 1.0.1 in Peres). Each move consists of a player placing an X or an O in some empty location on the board. The game ends when either there are no empty locations on the board, or there are 3 X's or O's in a single row, column, or diagonal. Since there are only 9 total positions on the board, and placing an X or an O reduces the number of empty locations by 1, we have $B(x) \leq 9$ for all starting positions x. Thus Tic-tac-toe is progressively bounded.

From presonal experience, both players have a strategy forcing at least a draw. □

Exercise 1.7. Let n be a positive integer. Consider the game Chomp played on an $n \times n$ board. Explicitly describe the winning strategy for the first player. (Hint: the first move should remove the square which is diagonally adjacent to the lower left corner.)

Solution. Denote the square in the in the i-th row and j-th column by (i, j) so that $(1, 1)$ is in the lower left corner and (n, n) is in the upper right corner. Following the hint, the first player chomps at $(2, 2)$ so that the remaining board consists of an $n \times n$ "L"-shape. Note that the second player's move must be of the form $(1, i)$ or $(i, 1)$ since all remaining squares are of this form. If the second player's move is $(1, i)$, the first player responds with $(i, 1)$. Similarly, if the second player's move is $(i, 1)$, the first player responds with $(1, i)$. Thus the first player aways mimics the first player's previous move but on the opposite side of the L-shape. Notice that after the first player's move, the board is always an $i \times i$ L-shape. Since Chomp is progressively bounded (or by induction) the board will eventually consist of the single square $(1,1)$ after the first player's turn, which is a winning position for the first player. $\square$

Exercise 1.8. Consider the game of Chomp played on a board of size $2 \times \infty$. Recall that a typical Chomp game board is $n \times m$, so that the board has n rows and m columns. We can label the rows as $\{1, 2, \ldots, n\}$ and we can label the columns as $\{1, 2, \ldots, m\}$, where n, m are positive integers. On a $2 \times \infty$ board, we label the rows as $\{1, 2\}$, and we label the columns as $\{1, 2, 3, 4, 5, 6, \ldots\}$. We can think of the row and column labels as coordinates in the xy-plane. So, the lower left corner will still have x-coordinate 1 and y-coordinate 1, so that the lower left square has coordinates $(1,1)$; the square to the right of this has coordinates $(2,1)$, and so on.

On the $2 \times \infty$ board, which player has a winning strategy? Prove your assertion, and describe explicitly the winning strategy.

Let $n > 2$ be an integer. On the $n \times \infty$ board, which player has a winning strategy? Prove your assertion, and describe explicitly the winning strategy.

Let $n > 2$ be an integer. On the $\infty \times \infty$ board, which player has a winning strategy? Prove your assertion, and describe explicitly the winning strategy.

Solution. On the $2 \times \infty$ board, the previous (second) player has a winning strategy. As before, denote the square in the i-th row and j-th column by (i, j). Suppose the first player's first move is $(1, i)$ for some i. The resulting position is a $2 \times i$ board. By Theorem 1.1.2 in Peres, the next player (i.e., second player) then has a winning strategy from this position.

Now suppose the first player's first move is of the form $(2, i)$. We will show that from the resulting position, the next player (second player) still has a winning strategy. Specifically we'll show that chomping $(1, i+1)$ is a winning move. The result follows from the following claim:

Claim. Let C_i be the position in which the top row contains i squares, and the bottom row contains $i+1$ squares (i.e., C_i is the $2 \times i+1$ rectangle with the upper right corner removed. Then the previous player has a winning strategy from C_i.

We will prove the claim by induction on i. For the base case $i = 1$, C_1 is the 2×2 rectangle with the upper right square removed. It is easy to verify that the previous player wins from this position (the next player has only 2 possible moves). For the inductive step, suppose the previous player has a winning strategy for C_j for all $1 < j < i$. From C_i, the next player's move must be of the form $(1, j)$ or $(2, j)$ for some j. In the first case, the resulting position is a $2 \times j$ rectangle hence the following player (i.e., the previous player from C_i) has a winning strategy by Theorem 1.1.2 in Peres. On the other hand, if the next move is $(2, j)$ then the previous player can move $(1, j+1)$ resulting in position C_j. By the inductive hypothesis, the previous player has a winning strategy from C_j. In either case, the previous player wins, so the claim follows.

Using the claim, we can explicitly describe a winning strategy for the second player:
(1) If first player chomps $(1, i)$, second player chomps $(2, i-1)$.
(2) If first player chomps $(2, i)$, second player chomps $(1, i+1)$.
(3) Repeat until the second player wins.

Since we've shown that the previous player has a winning strategy for the $2 \times \infty$ board, we can construct a winning strategy for the next player (i.e., first player) for the $n \times \infty$ and $\infty \times \infty$ boards. In either case, the first player's opening move is $(3, 1)$ so the resulting position is the $2 \times \infty$ position. After this opening move, the previous player (i.e., the first player) has a winning strategy.

□

2. Homework 2

Exercise 2.1. Compute the following nim-sums: $3 \oplus 4, 5 \oplus 9$. Then, let a, b, c be nonnegative integers. Prove that $a \oplus a = 0$ and $(a \oplus b) \oplus 0 = a \oplus b$.

Solution. Write $3 = 011_2, 4 = 100_2, 5 = 0101_2, 9 = 1001_2$. Thus

$$3 \oplus 4 = 011_2 \oplus 100_2 = 111_2 = 7$$

and

$$5 \oplus 9 = 0101_2 \oplus 1001_2 = 1100_2 = 12.$$

For the second part, suppose a and b can be expressed in binary as $a = a_k a_{k-1} \cdots a_0$ and $b = b_k b_{k-1} \cdots b_0$, respectively. Since for any bit a_i, we have $a_i \oplus a_i = 0$ (because $0 \oplus 0 = 1 \oplus 1 = 0$) we compute

$$\begin{aligned} a \oplus a &= (a_k a_{k-1} \cdots a_0) \oplus (a_k a_{k-1} \cdots a_0) \\ &= (a_k \oplus a_k)(a_{k-1} \oplus a_{k-1}) \cdots (a_0 \oplus a_0) \\ &= 00 \cdots 0 \\ &= 0. \end{aligned}$$

Similarly (using the fact that for any bit a_i, we have $a_i \oplus 0 = a_i$) we compute

$$\begin{aligned} (a \oplus b) \oplus 0 &= ((a_k a_{k-1} \cdots a_0) \oplus (b_k b_{k-1} \cdots b_0)) \oplus (00 \cdots 0) \\ &= ((a_k \oplus b_k)(a_{k-1} \oplus b_{k-1}) \cdots (a_0 \oplus b_0)) \oplus (00 \cdots 0) \\ &= ((a_k \oplus b_k) \oplus 0)((a_{k-1} \oplus b_{k-1}) \oplus 0) \cdots ((a_0 \oplus b_0) \oplus 0) \\ &= (a_k \oplus b_k)(a_{k-1} \oplus b_{k-1}) \cdots (a_0 \oplus b_0) \\ &= a \oplus b. \end{aligned}$$

□

Exercise 2.2. Consider the nim position $(9, 10, 11, 12)$. Which player has a winning strategy from this position, the next player or the previous player? Describe the winning first move.

Solution. Recall that by Bouton's Theorem (Theorem 1.1.3 in Peres), the previous player has a strategy if and only if the nim sum of the components is 0. Writing $9 = 1001_2, 10 = 1010_2, 11 = 1011_2$, and $12 = 1100_2$, we compute the nim sum

$$\begin{array}{cccc} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \end{array}$$

so that $9 \oplus 10 \oplus 11 \oplus 12 = 4 \neq 0$. Thus the next player has a winning strategy. The winning move is to remove 4 from the last pile. □

Exercise 2.3. Let G_1, G_2 be games. Let x_i be a game position for G_i, and let N_{G_i}, P_{G_i} denote, N and P respectively for the game G_i, for each $i \in \{1, 2\}$. Show the following:
(i) If $x_1 \in P_{G_1}$ and if $x_2 \in P_{G_2}$, then $(x_1, x_2) \in P_{G_1+G_2}$.
(ii) If $x_1 \in P_{G_1}$ and if $x_2 \in N_{G_2}$, then $(x_1, x_2) \in N_{G_1+G_2}$.
(iii) If $x_1 \in N_{G_1}$ and if $x_2 \in N_{G_2}$, then (x_1, x_2) could be in either $N_{G_1+G_2}$ or $P_{G_1+G_2}$.

Solution. Recall that given two games G_1 and G_2, the sum $G_1 + G_2$ is the game in which during each turn a player chooses which game G_i to play. Terminal positions are positions of the form (t_1, t_2) where each t_i is terminal for G_i.

We will prove (i) and (ii) simultaneously by induction on $n = B(x_1, x_2)$, the length of the longest game in $G_1 + G_2$ starting at position (x_1, x_2). For the base case $n = 0$, then (x_1, x_2) is a terminal position, hence $x_i \in P_{G_i}, x_2 \in P_{G_2}$, and $(x_1, x_2) \in P_{G_1+G_2}$. Therefore (i) and (ii) hold when $n = 0$. For the inductive step suppose (i) and (ii) hold for all y_1, y_2 with $B(y_1, y_2) \leq n - 1$. Suppose $x_i \in P_{G_i}$ for $i = 1, 2$ and $B(x_1, x_2) = n$. Suppose the next player chooses to play G_1 and moves to position y_1. By the definition of P_{G_1}, we have $y_1 \in N_{G_1}$, hence we have $(y_1, x_2) \in N_{G_1+G_2}$ by the inductive hypothesis. Thus, the previous player from (x_1, x_2) has a winning strategy. An identical argument shows that the previous player from (x_1, x_2) if the next player chooses to play G_2.

Now suppose $x_1 \in P_{G_1}$ and $x_2 \in N_{G_2}$. Then the next player can choose to play G_2 for which she has a winning move from x_2. Let y_2 be the resulting position. Then $y_2 \in P_{G_2}$, so that $(x_1, y_2) \in P_{G_1+G_2}$ by the inductive hypothesis. Thus, the next player (from position (x_1, x_2)) has a winning strategy so that $(x_1, x_2) \in N_{G_1+G_2}$. Therefore (i) and (ii) hold by induction.

For (iii), we will give explicit examples that demonstrate the two cases. Let G_1 and G_2 both be nim. First consider the case $x_1 = x_2 = (1)$, the nim position with one pile containing one chip. Then we have $x_1, x_2 \in N$. Notice that the position $(1, 1)$ in $G_1 + G_2$ is the same as the position $(1, 1)$ in ordinary nim. Thus, $(1, 1) \in P$. On the other hand, taking $x_1 = 1$ and $x_2 = 2$, we have $(1, 2) \in N$ while $(x_1), (x_2) \in N$. □

Exercise 2.4. Let G_1, G_2, G_3 be games. Show that the notion of two games being equivalent is an equivalence relation. That is, show the following

- G_1 is equivalent to G_1.
- If G_1 is equivalent to G_2, then G_2 is equivalent to G_1.
- If G_1 is equivalent to G_2, and if G_2 is equivalent to G_3, then G_1 is equivalent to G_3.

Solution. Recall that two games G_1 and G_2 with positions x_1 and x_2 are equivalent if for all games G and positions x, the outcome of (x_1, x) in $G_1 + G$ is the same as the outcome of (x_2, x) in $G_2 + G$.

To see that G_1 with position x_1 is equivalent to itself (equivalence is *reflexive*) note that (x_1, x) clearly has the same outcome as (x_1, x). Now suppose G_1 is equivalent to G_2. Since for all games G and positions x, (x_1, x) has the same outcome as (x_2, x), it is also the case that (x_2, x) has the same outcome as (x_1, x) (*symmetry*). Finally, Suppose that G_1 is equivalent to G_2 and G_2 is equivalent to G_3. Then for all G, (x_1, x) has the same outcome as (x_2, x), which has the same outcome as (x_3, x). In particular, (x_1, x) has the same outcome as (x_3, x) so that G_1 and G_3 are equivalent (*transitivity*).

□

Exercise 2.5. Show that in the game of chess, exactly one of the following situations is true:

(1) White has a winning strategy.
(2) Black has a winning strategy.
(3) Each of the two players has a strategy guaranteeing at least a draw.

You may assume that chess is progressively bounded. (Hint: you should not really need to use anything special about chess, other than that it is an impartial combinatorial game that is progressively bounded. Also, as usual, it is probably beneficial to start from a terminal position, and then work backwards, using induction.)

Solution. Let x be a position and i be a player (either black or white). Let $B(x,i)$ be the maximum number of moves of a game of chess starting from (x,i). That is, the board is in position x and i has the next move. We argue by induction on $B(x,i)$. For the base case, $B(x,i)=0$, so (x,i) is a terminal position. The terminal position corresponds either to a win for white, a win for black, or a draw.

For the inductive step, suppose that the claim (exactly one of items 1–3 above holds) is true for all (y,j) with $B(y,j)\leq n$. We will show that if $B(x,i)=n+1$, then exactly one of 1–3 holds for (x,i) as well. We assume that $i=$ white. The case where i is black is analogous. Suppose white can move from x to position y. Then $B(y,\text{black})\leq n$, so by the inductive hypothesis, one of the three possibilities holds.

(1) white has a winning strategy from (y,black).
(2) Black has a winning strategy from (y,black).
(3) Each player has a strategy from (y,black) resulting in a at least a draw.

Suppose that for every legal move $x\to y$ for white, the resulting position (y,black) satisfies (b) above. Then every move from (x,white) results in a winning strategy for black, hence black has a winning strategy from (x,white).

If not every legal move $x\to y$ for white satisfies (b), then there exists a move $x\to y$ for white such that (y,black) satisfies either (a) or (c). In the former case, white has a winning strategy starting from (x,white). On the other hand, if no move to (x,white) satisfies (a) but some move satisfies (c), then white has a move resulting in at least a draw. Thus, starting from (x,white), one of 1-3 above are satisfied. Therefore, by induction, every position satisfies one of 1-3 above, which gives the desired result. □

Exercise 2.6. We first describe the game of Y. In this game, there is an arrangement of white hexagons in an equilateral triangle. One player is assigned the color blue, and the other player is assigned the color yellow. The players then take turns filling in one hexagon at a time of their assigned color. The goal is to create a Y-shape that connects all three sides of the triangle. That is, the goal of the game is to have an unbroken path of a single color of hexagons that touches all three sides of the triangle.

Prove that the game of Hex can be realized as a special case of the game of Y. That is, the opening position on a standard hex board is equivalent to a particular game position in the game of Y. (Recall that we defined a notion for two games being equivalent.)

Solution. Consider the initial configuration for Y below. Notice that blue and yellow both

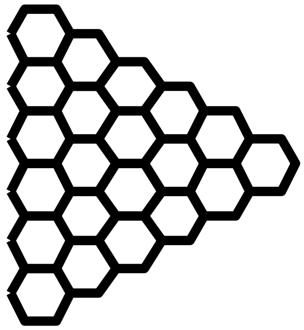
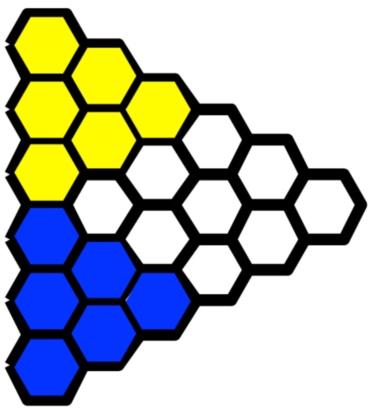


Figure 1. A Starting Position in the game of Y



already touch two sides of the triangle, and the remaining region is an $n \times n$ parallelogram. Thus the remaining positions constitute an initial position for hex. Further, if blue wins the

$\square$

Exercise 2.7. Describe the optimal strategies for both players in rock-paper-scissors. Prove that these strategies are optimal. This game is described by the following payoff matrix.

		Player II		
		R	P	S
Player I	R	0	-1	1
	P	1	0	-1
	S	-1	1	0

Solution. We claim that the optimal strategies for rock-paper-scissors are

$$\vec{x} = \vec{y} = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}.$$

To show that $\vec{x}$ is Player I's optimal strategy, we must show that

$$\min_{\vec{y} \in \Delta_3} \vec{x}^T A \vec{y} = \max_{\vec{x} \in \Delta_3} \min_{\vec{y} \in \Delta_3} \vec{x}^T A \vec{y}$$

where Δ_3 is the set of all probability vectors of length 3.

Where A is the payoff matrix

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

where

$$\Delta_3 = \{(z_1, z_2, z_3) \in \mathbb{R}^3 | z_i \geq 0, z_1 + z_2 + z_3 = 1\}$$

and

We claim that for any fixed $\vec{x} \in \Delta_3$, we have

$$\min_{\vec{y} \in \Delta_3} \vec{x}^T A \vec{y} \leq 0.$$

To see this, observe that for $\vec{x} = (x_1, x_2, x_3)$, we have

$$\vec{x}^T A = (x_2 - x_3 \quad x_3 - x_1 \quad x_1 - x_2).$$

In particular, the sum of the entries of $\vec{x}^T A$ is 0, so at least one entry is non-positive. Suppose $x_2 - x_3 \leq 0$. Then by choosing $\vec{y} = (1, 0, 0)$, we have $\vec{x}^T A \vec{y} \leq 0$. Thus $\min_{\vec{y}} \vec{x}^T A \vec{y} \leq 0$. A similar argument shows that for any $\vec{x}^T$, we have $\min_{\vec{y}} \vec{x}^T A \vec{y} \leq 0$ for every $\vec{x}$. This implies that

$$\max_{\vec{x}} \min_{\vec{y}} \vec{x}^T A \vec{y} \leq 0. \quad (*)$$

Choosing $\vec{x} = (1/3, 1/3, 1/3)$ we obtain

$$\vec{x}^T A = (0, 0, 0) \quad \text{hence} \quad \min_{\vec{y}} \vec{x}^T A \vec{y} = 0.$$

By (*), this final expression implies that

$$\min_{\vec{y}} \vec{x}^T A \vec{y} = 0 \geq \max_{\vec{x}} \min_{\vec{y}} \vec{x}^T A \vec{y}$$

so that $\vec{x}$ is indeed an optimal strategy for player I for rock-paper-scissors. The argument that $\vec{y} = (1/3, 1/3, 1/3)$ is optimal for player II is analogous. Using these strategies, we can see that the value of rock-paper-scissors is 0. □

Exercise 2.8. Describe the optimal strategies for both players for the two-person zero-sum game described by the payoff matrix Prove that these strategies are optimal.

Player I \ Player II	A	B
C	0	2
D	4	1

Solution. We claim that the optimal strategies for Player I and Player II are

$$\vec{x} = \begin{pmatrix} 3/5 \\ 2/5 \end{pmatrix} \quad \text{and} \quad \vec{y} = \begin{pmatrix} 1/5 \\ 4/5 \end{pmatrix}.$$

To prove the optimality of $\vec{x}$ we must show that

$$\min_{\vec{y} \in \Delta_2} \vec{x}^T A \vec{y} = \max_{\vec{x} \in \Delta_2} \min_{\vec{y} \in \Delta_2} \vec{x}^T A \vec{y}$$

where A is the payoff matrix

$$A = \begin{pmatrix} 0 & 2 \\ 4 & 1 \end{pmatrix}$$

□

Solution. First observe that

$$(3/5 \quad 2/5) \begin{pmatrix} 0 & 2 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 1/5 \\ 4/5 \end{pmatrix} = 8/5,$$

so the purported value of the game defined by A is 8/5. For any $\vec{x} \in \Delta_2$ we can write $\vec{x} = (p, 1-p)$, and similarly any $\vec{y}$ can be written $\vec{y} = (q, 1-q)$. Thus, we can write

$$\begin{aligned} \vec{x}^T A \vec{y} &= (p \quad 1-p) \begin{pmatrix} 0 & 2 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} q \\ 1-q \end{pmatrix} = (p \quad 1-p) \begin{pmatrix} 2-2q \\ 4q+1 \end{pmatrix} \\ &= 2pq - 2q + 4q - 4pq + p + 1 - q = (4-4p)q + (p+1)(1-q). \end{aligned}$$

Thus, for any fixed p, we can compute

$$\min_{\vec{y} \in \Delta_2} \vec{x}^T A \vec{y} = \min_{q \in [0,1]} (4-4p)q + (p+1)(1-q).$$

Since $0 \leq q \leq 1$ and we are holding p is constant, this expression takes on a minimum value of $\min\{4-4p, p+1\}$ with $q = 1$ or $q = 0$. In order to maximize this expression, Player I must choose p so that $4 - 4p = p + 1$ which gives $p = 3/5$. Therefore, we obtain

$$\max_{\vec{x}} \min_{\vec{y}} \vec{x}^T A \vec{y} = \max_{p} \min_{q} (4-4p)q + (p+1)(1-q) = 8/5,$$

and the maximum occurs when $p = 3/5$ (i.e., $\vec{x} = (3/5, 2/5)$). A similar argument shows that

$$\min_{\vec{y}} \max_{\vec{x}} \vec{x}^T A \vec{y} = \min_{q} \max_{p} (2-2q)p + (3q+1)(1-p) = 8/5$$

where the minimum occurs when $q = 1/5$ (i.e., $\vec{y} = (1/5, 4/5)$). □

3. Homework 3

Exercise 3.1. This exercise deals with subsets of the real line. Show that [0, 1] is closed, but (0, 1) is not closed.

Solution. Recall that a set K is closed if and only if for every convergent sequence $x^{(1)}, x^{(2)}, ...$ such that $x^{(i)} \in K$ for all j, we have $x = \lim_{j\to\infty} x^{(i)} \in K$. Therefore, to show that [0, 1] is closed, we must show that every convergent sequence $\{x^{(i)}\}$ such that $0 \leq x^{(i)} \leq 1$ for all j has $x = \lim_{j\to\infty} x^{(i)}$ with $0 \leq x \leq 1$. We will argue by contradiction that $x \leq 1$ by contradiction – the proof that $0 \leq x$ is similar. Suppose towards a contradiction that $1 < x$. Since $x = \lim x^{(i)}$, we know that for every $\varepsilon > 0$, there exists N such that $j > N$ implies $|x - x^{(i)}| < \varepsilon$. However, $x^{(i)} \leq 1 < x$ implies that $|x^{(i)} - x| \geq |x - 1| > 0$. Thus, choosing $\varepsilon = |x-1|/2$ contradicts that $x = \lim x^{(i)}$.

To show that (0, 1) is not closed, we must find a convergent sequence $\{x^{(i)}\}$ such that $x^{(i)} \in (0,1)$ for all j, but with $x = \lim_{j\to\infty} x^{(i)} \notin (0,1)$. To this end consider the sequence $x^{(i)} = \frac{1}{j+1}$. Note that $x^{(i)} \in (0,1)$for all j, but $x = \lim_{j\to\infty} \frac{1}{j+1} = 0 \notin (0,1)$. □

Exercise 3.2. This exercise deals with subsets of Euclidean space R^d where $d \geq 1$. Show that the intersection of two closed sets is a closed set.

Solution. Suppose K_1 and K_2 are closed. To see that $K = K_1 \cap K_2$ is closed, suppose $\{x^{(i)}\} \subseteq K$ is a convergent sequence with $x = \lim x^{(i)}$. We must show that $x \in K$. To this end, note that since K_1 is closed and $x^{(i)} \in K \subseteq K_1$ for all j, we have $x \in K_1$. Similarly, $x \in K_2$, hence $x \in K_1 \cap K_2 = K$. Therefore, K is closed. □

Exercise 3.3. Define $f : R^d \to R$ by $f(x) := ||x||$. Show that f is continuous. (Hint: you may need to use the triangle inequality, which says that $||x + y|| \leq ||x|| + ||y||$, for any $x, y \in R^d$.)

Solution. Recall that $f : K \to R$ is continuous if for every convergent sequence $\{x^{(i)}\} \subseteq K$ with $x = \lim x^{(i)}$, we have $\lim f(x^{(i)}) = f(x)$. For $f(x) = ||x||$, we have

$f(x^{(i)}) = ||x^{(i)}|| = ||x^{(i)} - x + x||$

$\leq ||x^{(i)} - x|| + ||x||.$

Since $\lim ||x^{(i)} - x|| = 0$ (indeed, this is what it means for $x = \lim x^{(i)}$) we have $\lim f(x^{(i)}) = ||x|| = f(x)$, as desired. □

Exercise 3.4. Describe in words the set of points (x_1, x_2) in the plane such that $(x_1, x_2) \geq (3, 4)$.

Solution. Recall that $(x, y) \geq (x', y')$ if and only if $x \geq x'$ and $y \geq y'$. Thus $(x_1, x_2) \geq (3, 4)$ if and only if $x_1 \geq 3$ and $x_2 \geq 4$. Both of these equations are satisfied precisely for quadrant above and to the right of $(3, 4)$ in the plane. □

Exercise 3.5. Let d be a positive integer. Consider

$\Delta_d := \{x = (x_1, ..., x_d) \in R^d : \sum_{i=1}^d x_i = 1, x_i \geq 0, \forall 1 \leq i \leq d\}.$

Prove that Δ_d is convex, closed and bounded.

Solution. Recall that a set K is convex if for every $x, y \in K$ and for every $t \in [0, 1]$, we have $tx + (1-t)y \in K$. Suppose $x, y \in \Delta_d$, and write $x = (x_1, ..., x_d), y = (y_1, ..., y_d)$. Fix $t \in [0, 1]$ and define

$z = tx + (1 - t)y = (tx_1 + (1 - t)y_1, ..., tx_d + (1 - t)y_d) = (z_1, ..., z_d).$

To see that $z \in \Delta_d$, first note that because $x_i, y_i, t, 1 - t \geq 0$ for all i, we have $z_i = tx_i + (1 - t)y_i \geq 0$ for all i. Further

$\sum_{i=1}^d z_i = \sum_i (tx_i + (1 - t)y_i) = t \sum_i x_i + (1 - t) \sum_i y_i = t + 1 - t = 1.$

Thus $z \in \Delta_d$ as desired.

To see that Δ_d is closed, first observe that the function $f : R^d \to R$ defined by $f(x_1, ..., x_d) = x_1 + ... + x_d$ is continuous. Therefore, if $\{x^{(i)}\}$ is a convergent sequence in Δ_d with $x = \lim x^{(i)}$, we have $\lim f(x^{(i)}) = f(x)$. Further, for all $x^{(i)}$ we have $f(x^{(i)}) = 1$, hence $f(x) = \lim 1 = 1$. We still need to show that if $x = (x_1, ..., x_d)$, then $x_i \geq 0$ for all i. To this end, observe that if $x^{(j)} = (x_1^{(j)}, x_2^{(j)}, ..., x_d^{(j)})$, then the sequence $x_1^{(1)}, x_1^{(2)}, x_1^{(3)}, ...$ is a convergent sequence in [0, 1] with $x_i = \lim x_i^{(j)}$. Thus, by exercise 1, we have $x_i \in [0, 1]$, hence $x_i \geq 0$.

Finally, to show that Δ_d is bounded, notice that for $x = (x_1, ..., x_d) \in \Delta_d$, we have $x_i \leq 1$ for all i. Therefore,

$||x|| = \sqrt{x_1^2 + ... + x_d^2} \leq \sqrt{1^2 + ... + 1^2} = \sqrt{d}.$

Therefore, $\Delta_d \subseteq B(\sqrt{d})$, the ball of radius $\sqrt{d}$ centered at the origin. Thus Δ_d is bounded. □

Exercise 3.6.

- Let K be the set of points (x, y) in the plane such that $|x| + |y| \leq 2$. Is K convex? Prove your assertion.
- Let K be the set of points (x, y, z) in R^3 such that $max(|x|, |y|, |z|) \leq 1/2$. Is K convex? Prove your assertion.
- Let K be the set of points (x, y, z, w) in R^4 such that $x^2 + y^2 + z^2 + w^2 < 1$. You may assume that K is convex. Find a hyperplane that separates K from the point (0, 1, 1, 0).

Solution. For the first part, K is convex. To see this, suppose $(x, y), (z, w) \in K$. That is, $|x| + |y| \leq 2$ and $|z| + |w| \leq 2$. Then for $t \in [0, 1]$,

$|tx + (1-t)z| + |ty + (1-t)w| \leq t|x| + (1-t)|z| + t|y| + (1-t)|w|$

$= t(|x| + |y|) + (1-t)(|z| + |w|)$

$\leq t \cdot 2 + (1-t) \cdot 2$

$= 2.$

Thus, the first K is convex.

The second K is also convex. To see this, suppose $(x, y, z), (x', y', z') \in K$. We must show that for all $t \in [0, 1]$, we have $(tx + (1-t)x', ty + (1-t)y', tz + (1-t)z') \in K$. That is, for example, $|tx + (1-t)x'| \leq 1/2$. To see this note that

$|tx + (1-t)x'| \leq t|x| + (1-t)|x'| \leq max(|x|, |x'|) \leq 1/2.$

Similarly, $|ty + (1-t)y'| \leq 1/2$ and $|tz + (1-t)z'| \leq 1/2$, implying that the maximum of these three values is at most 1/2, as desired.

For the final part, notice that K is the unit ball in R^4. Since (0, 1, 1, 0) lies outside the ball, we expect to find a separating hyperplane. Let $x = (0, 1, 1, 0)$. Then for any $y = (x, y, z, w) \in K$, the Cauchy-Schwarz inequality gives

$x^T y = x \cdot y \leq ||x|| \cdot ||y|| \leq \sqrt{2}.$

On the other hand, $x^T(0, 1, 1, 0) = 2$. Thus for any c satisfying $\sqrt{2} < c < 2$, the hyperplane given by

$\{y \in R | x \cdot y = c\}$

is a separating hyperplane. □

Exercise 3.7. Show that the intersection of two convex sets is convex. Then, show that the intersection of any finite number of convex sets is convex. Finally, find two convex sets A, B such that the union $A \cup B$ is not convex.

Solution. Suppose A and B are convex, and let $x, y \in A \cap B$. We must show that for all $t \in [0, 1], tx + (1-t)y \in A \cap B$. To this end, note that $tx + (1-t)y \in A$ because A is convex, while $tx + (1-t)y \in B$ because B is convex. Thus $tx + (1-t)y \in A \cap B$, as desired.

Now suppose $A_1, A_2, ..., A_n$ are convex sets. We argue by induction on n that $A_1 \cap ... \cap A_n$ is also convex. The base case, $n = 1$ is clear because A_1 is assumed to be convex. For the inductive step, define

$B = A_1 \cap A_2 \cap ... \cap A_{n-1}.$

By the inductive hypothesis, B is convex. By the argument above (for two convex sets),

$B \cap A_n = A_1 \cap A_2 \cap ... \cap A_{n-1} \cap A_n$

is also convex, which gives the desired result.

For the final part, consider the example A = {0} and B = {1}. Then A and B are both convex, but AUB = {0, 1} is not convex. For example, $(1/2)0 + (1/2)1 = (1/2) \notin AUB$. □

Exercise 3.8. Let A be an $m \times n$ real matrix. Let $b \in R^m, c \in R^n$. Using the Minimax Theorem, prove the following equality, which is known as duality for linear programming:

$\max_{x \in R^n, Ax \leq b, x \geq 0} c^T x = \min_{y \in R^m, A^T y \leq c, y \geq 0} b^T y$

Consider now an example where n = m = 2, b = (1, 0), c = (1, 1) and A = $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Using the duality above, show that

$\max_{x \in R^n : Ax \leq b} c^T x \leq 1$.

Solution. Consider the two person zero sum game with $(m+n+1) \times (m+n+1)$ payoff matrix

$$B = \begin{pmatrix} 0 & A & -b \\ -A^T & 0 & c \\ b^T & -c^T & 0 \end{pmatrix}$$

First observe that B is antisymmetric – that is, $B^T = -B$. Therefore, the value of the game must be 0. To see this, suppose X and Y are optimal strategies for player I and II respectively. If $X^T BY \leq 0$, then player I can switch strategies to Y. Then the expected payoff $Y^T BY$ satisfies

$Y^T BY = (Y^T BY)^T = Y^T B^T (Y^T)^T = Y^T(-B)Y = -Y^T BY$,

implying that $Y^T BY = 0$. Thus $X^T BY < Y^T BY$, contradicting the optimality of X. A similar argument shows that $X^T BY$ cannot be positive, so it must be 0. Note that this

argument also shows that if Y is an optimal strategy for player II, Y is also an optimal strategy for player I. Now let

$$Y = \begin{pmatrix} y \\ x \\ t \end{pmatrix}$$

be an optimal strategy (for both players). By Exercise 10, the smallest row in BY is positive, hence we have

$$BY = \begin{pmatrix} 0 & A & -b \\ -A^T & 0 & c \\ b^T & -c^T & 0 \end{pmatrix} \begin{pmatrix} y \\ x \\ t \end{pmatrix} = \begin{pmatrix} Ax - bt \\ -A^T y + ct \\ b^T y - c^T x \end{pmatrix} \geq 0$$

We can rewrite the final inequality as

$Ax \geq bt$,

$A^T y \leq ct$,

$b^T y - c^T x \geq 0$.

As the problem indicates, assume that $t > 0$. Then we will show that x/t maximizes $c^T x$, while y/t minimizes $b^T y$ (while $A(x/t) \geq b$ and $A^T(y/t) \leq c$). To this end, we claim that for any $x \in R^m$ with $Ax \geq b$ and $y \in R^n$ with $A^T y \leq c$ and $x, y \geq 0$, we have $b^T y - c^T x \geq 0$. To see this, observe that

$Ax \geq b, y \geq 0 \implies b^T y \leq (Ax)^T y = x^T A^T y$.

Similarly,

$A^T y \leq c, x \geq 0 \implies c^T x \geq (A^T y)^T x = y^T Ax$.

Putting the previous two expressions together, we find

$b^T y \leq x^T A^T y = y^T Ax \leq c^T x$.

Therefore, $b^T y - c^T x \leq 0$. Since the points (x/t) and (y/t) satisfy the opposite inequality $(b^T(y/t) - c^T(x/t) \geq 0)$, we must have that, in fact, $b^T(y/t) = c^T(x/t)$ which is optimal.

For the particular example with

$b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

We have

$\max_{y \in R^2, A^T y \leq c, y \geq 0} b^T y = \min_{x \in R, Ax \geq b, x \geq 0} c^T x$

Thus, finding any $x \in R^2$ which satisfies $Ax \geq b$ and $x \geq 0$ gives us an upper bound for the max on the left side above. In particular, taking $x = (1, 0)$, we have $Ax \geq b, x \geq 0$ and $c^T x = 1$. Thus

$\max_{y \in R^2, A^T y \leq c, y \geq 0} b^T y \leq 1$

as desired. □

Exercise 3.9. Let $x \in \Delta_m, y \in \Delta_n$ and let A be an $m \times n$ matrix. Show that

$\max_{x \in \Delta_m} x^T A y = \max_{i=1,\ldots,m} (Ay)_i$,

$\min_{y \in \Delta_n} x^T A y = \min_{j=1,\ldots,n} (x^T A)_j$.

Using this fact, show that

$\min_{y \in \Delta_n} \max_{x \in \Delta_m} x^T A y = \min_{y \in \Delta_n} \max_{i=1,\ldots,m} (Ay)_i$.

$$\max_{x \in \Delta_m} \min_{y \in \Delta_n} x^T A y = \max_{x \in \Delta_m} \min_{j=1,\ldots,n} (x^T A)_j.$$

Using the second equality, conclude that the value of the game with payoff matrix A can be found via the following Linear Programming problem:

Maximize t subject to the constraints: $\sum_{i=1}^{n} A_{ij} x_i \geq t$, for all $1 \leq i \leq m$; $\sum_{j=1}^{m} x_i = 1$; $x \geq (0, ..., 0)$.

Efficient methods for solving linear programming problems are well-known. However, below we will focus on ways to compute the values of two-person zero-sum games by hand.

Solution.

We first show that

$$\max_{x \in \Delta_m} x^T A y = \max_{i=1,\ldots,m} (Ay)_i.$$

The proof of $\min_{y \in \Delta_n} x^T A y = \min_{j=1,\ldots,n} (x^T A)_j$ is analogous. We argue by induction on m. The base case, $m = 1$ is trivial as in this case Ay consists of a single entry and $\Delta_m = \{1\}$. Assume that the statement is true for $m - 1$, and suppose $\bar{x} = \bar{x}$ maximizes $x^T A y$. Denote $x = (x_1, x_2, ..., x_m)$. If any $x_i = 0$, then we have $x \in \Delta_{m-1}$, hence the claim follows by induction. Now assume that $x_i > 0$ for all i. Let k be the index for which $(Ay)_k$ is maximal. Then for any $i \neq k$, we can form the vector

$$x' = (x_1, ..., x_{i-1}, 0, x_{i+1}, ..., x_{k-1}, x_k + x_i, x_{k+1}, ..., x_m)$$

and we have

$$x' A y \geq \bar{x} A y = \max_{x \in \Delta_m} x^T A y.$$

Further, $x' \in \Delta_{m-1}$ because one of its entries is 0. Thus, by the inductive hypothesis, there exists k such that $x = e_k$ maximizes $x^T A y$ where e_k is the vector with a 1 in its k-th coordinate and 0's elsewhere.

Since for all y we have

$$\max_{x\in\Delta_m} x^T Ay = \max_{i=1,\dots,m} (Ay)_i$$

taking a minimum over all values of y gives

$$\min_{y\in\Delta_n} \max_{x\in\Delta_m} x^T Ay = \min_{y\in\Delta_n} \max_i (Ay)_i.$$

Similarly,

$$\max_{x\in\Delta_m} \min_{y\in\Delta_n} x^T Ay = \max_{x\in\Delta_m} \min_{j=1,\dots,n} (x^T A)_j.$$

Suppose $A = (a_{ij})$, $x = (x_1, ..., x_m)$, and $y = (y_1, ..., y_n)$. Then we can express

$$x^T Ay = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j,$$

hence an optimal strategy for the row player is the choice of $x = \bar{x}$ for which

$$\max_{x\in\Delta_m} \min_{y\in\Delta_n} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j = \min_{y\in\Delta_n} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j.$$

By the equality above, this last expression is equal to

$$\max_{x} \min_{j=1,\dots,n} \sum_{i=1}^{m} (a_{ij} x_i)_j.$$

Let $t = \max_x \min_j \sum_i (A_{ij} x_i)_j$. Then the previous expression implies that $x \in \Delta_m$ is an optimal strategy if and only if

$$\sum_{i=1}^{m} a_{ij} x_j \geq t \text{ for all } j.$$

Thus finding an optimal strategy for x is equivalent to maximizing t subject to $\sum a_{ij} x_i \geq t$ for all i, $\sum x_i = 1$ and $x \geq (0, ..., 0)$, as desired. □

Exercise 3.10. Find the value of the two-person zero-sum game described by the payoff matrix

$$A = \begin{pmatrix} 0 & 9 & 1 & 1 \\ 5 & 0 & 6 & 7 \\ 2 & 4 & 3 & 3 \end{pmatrix}$$

Solution. We use the technique of domination to reduce the dimension of the payoff matrix. Let C_1, C_2, C_3, C_4 denote the columns of A. Notice that $C_3 < C_4$, hence no optimal strategy $\vec{y} = (y_1, y_2, y_3, y_4)$ for player II will have $y_4 > 0$. Thus we can eliminate the final column. The game is thus equivalent to one with payoff matrix

$$\begin{pmatrix} 0 & 9 & 1 \\ 5 & 0 & 6 \\ 2 & 4 & 3 \end{pmatrix}$$

Let R_1, R_2, R_3 denote the rows of this matrix. Since $R_3 < \frac{1}{4}R_1 + \frac{1}{4}R_2$, no optimal strategy $\vec{x} = (x_1, x_2, x_3)$ for player I will have $x_3 > 0$. Thus the game is equivalent to one with payoff matrix

$$\begin{pmatrix} 0 & 9 & 1 \\ 5 & 0 & 6 \end{pmatrix}$$

Let D_1, D_2, D_3 denote the columns of this matrix. Then $\frac{5}{9}D_1 + \frac{4}{9}D_2 < D_3$, so no optimal strategy will have $y_3 > 0$. Therefore, the game is equivalent to one with payoff matrix

$$\begin{pmatrix} 0 & 9 \\ 5 & 0 \end{pmatrix}$$

To find the value of this game, suppose $\vec{x} = (p, 1-p)$ and $\vec{y} = (q, 1-q)$. An optimal strategy for I maximizes $\min_{\vec{y}} \vec{x}^T A\vec{y}$. We compute

$$\vec{x}^T A\vec{y} = \begin{pmatrix} p & 1-p \end{pmatrix} \begin{pmatrix} 0 & 9 \\ 5 & 0 \end{pmatrix} \begin{pmatrix} q \\ 1-q \end{pmatrix} = 5(1-p)q + 9p(1-q).$$

The max-min is therefore achieved when $5(1-p) = 9p$, implying that $p = 5/14$. In this case, the value of the matrix is

$$\begin{pmatrix} 5/14 & 9/14 \end{pmatrix} \begin{pmatrix} 0 & 9 \\ 5 & 0 \end{pmatrix} \begin{pmatrix} q \\ 1-q \end{pmatrix} = \frac{45}{14}.$$

□

Exercise 3.11. Find the value of the two-person zero-sum game described by the payoff matrix

$$\begin{pmatrix} 0 & 7 & 0 & 6 \\ 4 & 4 & 3 & 3 \\ 8 & 2 & 6 & 0 \end{pmatrix}$$

Solution. Again, we use domination.

$$\begin{pmatrix} 0 & 7 & 0 & 6 \\ 4 & 4 & 3 & 3 \\ 8 & 2 & 6 & 0 \end{pmatrix} \xrightarrow{\begin{pmatrix} 4 & 4 & 3 & 3 \end{pmatrix} < \frac{1}{2}\begin{pmatrix} 0 & 7 & 0 & 6 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 8 & 2 & 6 & 0 \end{pmatrix}} \begin{pmatrix} 0 & 7 & 0 & 6 \\ 8 & 2 & 6 & 0 \end{pmatrix}$$

$$\xrightarrow{\begin{pmatrix} 0 & 6 \end{pmatrix} < \begin{pmatrix} 0 & 8 \end{pmatrix}, \begin{pmatrix} 6 & 0 \end{pmatrix} < \begin{pmatrix} 7 & 2 \end{pmatrix}} \begin{pmatrix} 0 & 6 \\ 6 & 0 \end{pmatrix}$$

The value of this final matrix (and hence the original matrix) is 3. □

Exercise 3.12. This Exercise shows that von Neumann's Minimax Theorem no longer holds when we consider games for three or more players.

First, note that there is a suitable generalization of this theorem to two-player general-sum games. That is if A is the payoff matrix for player I and B is the payoff matrix for player II, then

$$\max_{\vec{x}\in\Delta_m} \min_{\vec{y}\in\Delta_n} \vec{x}^T A\vec{y} = \min_{\vec{y}\in\Delta_n} \max_{\vec{x}\in\Delta_m} \vec{x}^T A\vec{y}.$$

$$\max_{\vec{x}\in\Delta_m} \min_{\vec{y}\in\Delta_n} \vec{x}^T B\vec{y} = \min_{\vec{y}\in\Delta_n} \max_{\vec{x}\in\Delta_m} \vec{x}^T B\vec{y}.$$

In words, the first equality says: the maximum over player I's strategies followed by the minimum of the other players strategies of the payoff of player I is equal to the minimum of the other players strategies followed by the maximum over player I's strategies of the payoff of player I.

Now, consider a three-player general-sum game. The analogue of von Neumann's Theorem just applied to player I would say: the maximum over player I's strategies followed by the minimum of the other players strategies of the payoff of player I is equal to the minimum of the other players strategies of followed by the maximum over player I's strategies of the payoff of player I.

Show that this statement is false for the following example.

	L	R
T	0	1
B	1	1

W

	L	R
T	1	1
B	1	0

E

These matrices describe the payoffs for player I. In the game, player I chooses a row (T or B), player II chooses a column (L or R), and player III chooses a matrix (W or E)

Solution. Suppose player's strategies are $\vec{x} = (p, 1-p)$, $\vec{y} = (q, 1-q)$, and $\vec{z} = (r, 1-r)$ respectively with $p, q, r \geq 0$. That is, player I chooses T with probability p, and B with probability $1-p$, and so on. We wish to show that

$$\max_{\vec{x}} \min_{\vec{y}} \min_{\vec{z}} f(\vec{x}, \vec{y}, \vec{z}) \neq \min_{\vec{y}} \min_{\vec{z}} \max_{\vec{x}} f(\vec{x}, \vec{y}, \vec{z})$$

where $f(\vec{x}, \vec{y}, \vec{z})$ is the payoff for player I. We first compute the right side of the expression above. To this end, suppose p and q are fixed. We can express the expected payouts for player I as

$$r \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} q \\ 1-q \end{pmatrix} + (1-r) \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q \\ 1-q \end{pmatrix} = \begin{pmatrix} r(1-q) + (1-r)1 \\ r + (1-r)q \end{pmatrix}$$

Given the payouts above, player I will choose $p = 0$ or 1 (whichever maximizes his payout). We claim that the largest entry in vector above is always at least $3/4$. To see this, suppose $r(1-q) + (1-r) = 1 - qr < 3/4$. We must show that the second coordante $r + (1-r)q = q + r - qr$ is always at least $3/4$ when the first coordinate satisfies $1 - qr \leq 3/4$. To this end, notice that the equations $q, r \leq 1$ and $1 - qr \leq 3/4$ (or equivalently $r \geq 1/4q$) form a bounded region in the qr-plane. Using multivariable calculus style optimization, the function $q + r - qr$ attains a minimum value of $3/4$ (at the point $p = q = 1/2$) in this region. Hence $r + q - rq \geq 3/4$ whenever $1 - rq \leq 3/4$, implying that

$$\min_{\vec{y}} \min_{\vec{z}} \max_{\vec{x}} f(\vec{x}, \vec{y}, \vec{z}) \geq 3/4.$$

On the other hand, suppose $\vec{x} = (p, 1-p)$ is fixed. Then

$$(p \quad 1-p) \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = (1-p \quad 1) \quad \text{and} \quad (p \quad 1-p) \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = (1 \quad p).$$

Thus, if p satisfies $p < 1/2$, players II and III can choose R and E respectively to obtain a payoff of $p < 1/2$. Similarly if $p > 1/2$, they can choose L and W to obtain a payoff of $1 - p < 1/2$. Thus

$$\max_{\vec{x}} \min_{\vec{y}} \min_{\vec{z}} f(\vec{x}, \vec{y}, \vec{z}) \leq 1/2.$$

□

4. Homework 4

Exercise 4.1. Show the following fact, which will be mentioned after our proof of Sperner's Lemma:

Let d be a positive integer. Let K be a closed and bounded subset of $\mathbb{R}^d$. Then the set $K \times K$ is also a closed and bounded set.

(Recall that $K \times K = \{(\vec{x}, \vec{y}) \in \mathbb{R}^d \times \mathbb{R}^d : \vec{x} \in K \text{ and } \vec{y} \in K\} \subseteq \mathbb{R}^{2d}$.)

Solution. First, we show that $K \times K$ is closed. To this end, we need to show that for any sequence $\vec{x}^{(1)}, \vec{x}^{(2)}, \ldots$ with $\vec{x}^{(i)} \in K \times K$ for all j, and $\vec{x} = \lim \vec{x}^{(i)}$, we have $\vec{x} \in K \times K$. Write $\vec{x}^{(i)} = (\vec{y}^{(i)}, \vec{z}^{(i)})$ with $\vec{y}^{(i)}, \vec{z}^{(i)} \in K$ and $\vec{x} = (\vec{y}, \vec{z})$. Notice that $\vec{y} = \lim \vec{y}^{(i)}$ and $\vec{z} = \lim \vec{z}^{(i)}$. Therefore, since K is closed, $\vec{y}, \vec{z} \in K$ implying that $\vec{x} = (\vec{y}, \vec{z}) \in K \times K$. Therefore, $K \times K$ is closed.

To see that $K \times K$ is bounded, suppose $K \subseteq B_n(r)$, the n dimensional ball of radius r centered at the origin. We must show that there exists some r' such that $K \times K \subseteq B_{2n}(r')$. To this end, we take $r' = 2r$. Then for any $\vec{x} = (\vec{y}, \vec{z}) \in K \times K$, we have $||\vec{y}||, ||\vec{z}|| \le r$. Therefore

$$\begin{aligned} ||\vec{x}|| &= ||(\vec{x}, \vec{y})|| \\ &= ||(\vec{x}, \vec{y}) - (0, \vec{y}) + (0, \vec{y})|| \\ &\le ||(\vec{x}, \vec{y}) - (0, \vec{y})|| + ||(0, \vec{y})|| \\ &= ||\vec{x}|| + ||\vec{y}|| \\ &< r + r. \end{aligned}$$

Thus, $K \times K$ is bounded, as desired. □

Exercise 4.2. Show the following facts, which will be used in our discussion of Correlated equilibria: For any $\vec{x}, \vec{y} \in \Delta_2$, $\vec{x}\vec{y}^T \neq \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$.

For any $\vec{x}, \vec{y} \in \Delta_2$, $\vec{x}\vec{y}^T \neq \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & 0 \end{pmatrix}$.

Solution. Write $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$. If the first equality held, we would have $x_1 y_2 = 0$, implying that $x_1 = 0$ or $y_2 = 0$. Since $x_1 y_1 = 1/2$, we cannot have $x_1 = 0$. However, we also have $y_2 x_2 = 1/2$, implying that $y_2 \neq 0$, a contradiction.

Similarly, if the second inequality held, we must have $x_2 y_2 = 0$, implying that $x_2 = 0$ or $y_2 = 0$. However, $x_2 y_1 = 1/3$ implying that $x_2 \neq 0$, while $x_1 y_2 = 1/3$ implying that $y_2 \neq 0$. Either way, we arrive at a contradiction.

This result also follows from the general result in linear algebra which states that if A is an $n \times m$ matrix, and B and $m \times k$ matrix, then rank(AB) ¡ rank(A), rank(B). Since $\vec{x}$ and $\vec{y}$ are 2×1 and 1×2 matrices, respectively, they both have rank at most 1. Thus, their product $\vec{x}\vec{y}^T$ must have rank at most 1. However, the two matrices above have rank 2! □

Exercise 4.3. Recall the prisoner's dilemma, which is described by the following payoffs

Recall that this two-person game has exactly one Nash equilibrium, where both parties confess. However, if this game is repeated an infinite number of times, or a random number of times, this strategy is no longer the only Nash equilibrium. This exercise explores the case where the game is repeated an infinite number of times. Let N be a positive integer.

		Prisoner II	
Prisoner I		silent	confess
	silent	$(-1,-1)$	$(-10,0)$
	confess	$(0,-10)$	$(-8,-8)$

Suppose the game is repeated infinitely many times, so that player I has payoffs $a_1, a_2, a_3, \ldots$ and player II has payoffs $b_1, b_2, b_3, \ldots$. That is, at the i^{th} iteration of the game, player I has payoff a_i and player II has payoff b_i. In the infinitely repeated game, each player would like to maximize her average payoff over time (if this average exists). That is, player I wants to maximize $\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} a_i$ and player II wants to maximize $\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} b_i$.

Consider the following strategy for player I. Suppose player I begins by staying silent, and she continues to be silent on subsequent rounds of the game. However, if player II confesses at round $i \geq 1$ of the game, then player I will always confess for every round of the game after round i. Player II follows a similar strategy. Suppose player II begins by staying silent, and she continues to be silent on subsequent rounds of the game. However, if player I confesses at round $j \geq 1$ of the game, then player II will always confess for every round of the game after round j.

Show that this pair of strategies is a Nash equilibrium. That is, no player can gain something by unilaterally deviating from this strategy.

Solution. Denote the strategy described above by $(\vec{x}, \vec{y})$. Observe that if both players follow this strategy, they will be silent in every round. Thus the payoff for player I in each round is -1, hence the average payoff for I is $a = -1$. Now suppose II plays $\vec{y}$, while I plays $\vec{x} \neq \vec{x}$. Since $\vec{x} \neq \vec{x}$, there is some round t in which I confesses. Since II uses the strategy $\vec{y}$ described above, II confesses in each round $i > t$. If II confesses in round i, the best payoff that I can achieve in that round is -8 (by confessing), hence we have $a_i \leq -8$ for all $i > t$. Thus

$$a = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} a_i \leq -8 < -1.$$

An identical argument shows that if II deviates from $\vec{y}$ while I plays $\vec{x}$ then II's average payoff is at most -8. Therefore, $(\vec{x}, \vec{y})$ is indeed a Nash equilibrium. □

Exercise 4.4. Show that the following strategy (known as "quid pro quo") is also a Nash equilibrium for Prisoner's Dilemma iterated an infinite number of times.

Player I begins by staying silent. If Player II plays x on round i, then Player I plays x on round $i+1$. Similarly, Player II begins by staying silent. If Player I plays x on round i, then Player II plays x on round $i+1$.

Solution. Notice that if both players follow the quid pro quo strategy $(\vec{x}, \vec{y})$ described above, then (by induction) in all rounds both players will be silent. Thus, the average payoff for both players is $a = b = -1$. Now suppose II plays $\vec{y}$ while I plays $\vec{x} \neq \vec{x}$, and define the sequence $c_1, c_2, \ldots$ by

$$c_i = \begin{cases} a_i & \text{if I is silent in rounds i and i - 1} \\ \frac{1}{2}(a_i + a_{i+1}) & \text{if I confesses in round i} \\ \frac{1}{2}(a_{i-1} + a_i) & \text{if I confesses in round i - 1 but not in round i} \end{cases}$$

Observe that

$$a = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} a_i = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} c_i$$

because each a_i appears either as some $c_i = a_i$ or $a_i/2$ appears in c_i and c_{i+1}. If I confesses in the i-th round, then II will confess in the $(i+1)$-st round, implying that a_{i+1} is at most -8. Thus, $c_i \le -4 \le -1$. In particular, $c_i \le -1$ for all i. Thus, if I deviates from $\vec{x}$, then we have

$$a = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} c_i \le -1 < \frac{1}{N} \sum_{i=1}^{N} -1 = -1.$$

A similar argument shows that $b \le -1$ if II deviates from $\vec{y}$ while I plays $\vec{x}$, which gives the desired result. □

Exercise 4.5. Find all Correlated Equilibria for the Prisoner's Dilemma.

Solution. Recall that the payoff matrices for the prisoner's dilemma are given by

$$A = \begin{pmatrix} -1 & -10 \\ 0 & -8 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 0 \\ -10 & -8 \end{pmatrix}$$

Suppose

$$Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}$$

is a correlated equilibrium. Thus, for player I, we must have

$$-z_{11} - 10z_{12} \ge 0z_{11} - 8z_{12} \qquad 0z_{12} - 8z_{22} \ge -z_{21} - 10z_{22}$$

and similarly for player II,

$$-z_{11} - 10z_{21} \ge 0z_{11} - 8z_{12} \qquad 0z_{12} - 8z_{22} \ge -z_{12} - 10z_{22}.$$

Rearranging gives

$$0 \ge z_{11} + 2z_{12} \qquad 0 \ge z_{12} + 2z_{22} \qquad 0 \ge z_{11} + 2z_{21} \qquad 0 \ge z_{12} + 2z_{22}.$$

Since $\sum_{i,j} z_{ij} = 1$ and $z_{ij} \ge 0$ for all i, j, these four inequalities combine to give $z_{22} = 1$, $z_{11} = z_{12} = z_{21} = 0$. Thus, the only correlated equilibrium is

$$Z = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

□

Exercise 4.6. Show that any convex combination of Nash equilibria is a Correlated Equilibrium. That is, if $z^{(1)}, \ldots, z^{(k)}$ are Correlated Equilibria, and if $t_1, \ldots, t_k \in [0,1]$ satisfy $\sum_{i=1}^{k} t_i = 1$, then $\sum_{i=1}^{k} t_i z^{(i)}$ is a Correlated Equilibrium.

Solution. We will show that the convex combination of two correlated equilibria is another correlated equilibrium. The general result follows by induction (on k). Suppose $z = (z_{ij})$ and $w = (w_{ij})$ are correlated equilibria for a game with payoff matrices A and B. Let $t \in [0,1]$.

We will show that $v = (v_{ij}) = tz + (1-t)w$ is also a correlated equilibrium. That is, we must show that for all $i, k \in \{1, 2, \ldots, m\}$

$$(*) \quad \sum_{j=1}^{n} v_{ij} a_{ij} \geq \sum_{j=1}^{n} v_{ij} a_{kj} \tag{2}$$

and for all $j, k \in \{1, 2, \ldots, n\}$

$$(**) \quad \sum_{i=1}^{n} v_{ij} b_{ij} \geq \sum_{i=1}^{n} v_{ij} b_{ik}. \tag{3}$$

We will prove (*) -the argument for (**) is analogous.

Since z and w are correlated equilibria, we have for all $i, k \in \{1, 2, \ldots, m\}$,

$$\sum_{j=1}^{n} z_{ij} a_{ij} \geq \sum_{j=1}^{n} z_{ij} a_{kj} \quad \text{and} \quad \sum_{j=1}^{n} w_{ij} a_{ij} \geq \sum_{j=1}^{n} w_{ij} a_{kj}.$$

Writing $v_{ij} = tz_{ij} + (1-t)w_{ij}$, we obtain

$$\begin{aligned}
\sum_{j=1}^{n} v_{ij} a_{ij} &= t \sum_{j=1}^{n} z_{ij} a_{ij} + (1-t) \sum_{j=1}^{n} w_{ij} a_{ij} \\
&\geq t \sum_{j=1}^{n} z_{ij} a_{kj} + (1-t) \sum_{j=1}^{n} w_{ij} a_{kj} \\
&= \sum_{j=1}^{n} (tz_{ij} + (1-t) w_{ij}) a_{kj} \\
&= \sum_{j=1}^{n} v_{ij} a_{kj}
\end{aligned}$$

which gives the desired result. □

Exercise 4.7. Recall the Game of Chicken is defined as follows. Each player chooses to chicken out (C) by swerving away, or she can continue drive straight (D). Each player would prefer to continue driving while the other chickens out. However, if both players choose to continue driving, catastrophe occurs. The payoffs follow:

		Player II C	Player II D
Player I	C	(6, 6)	(2, 7)
	D	(7, 2)	(0, 0)

Find all Nash equilibria for the Game of Chicken. Prove that these are the only Nash equilibria. Then, verify that

$$z = \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & 0 \end{pmatrix}$$

is a Correlated Equilibrium. Can you find a Correlated Equilibrium such that both players have a payoff larger than 5? (Hint: when trying to find such a matrix z, assume that $z_{22} = 0$ and $z_{12} = z_{21}$.)

Solution. We first observe that the pure strategies (C, D) and (D, C) are pure Nash equilibria, which correspond to the correlated equilibria

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

respectively. Recall that if $(\vec{x}, \vec{y})$ is a mixed Nash equilibrium, then Player I should be indifferent to his pure strategy payoffs for $A\vec{y}$. If $\vec{y} = (q, 1-q)$, then we compute

$$A\vec{y} = \begin{pmatrix} 6 & 2 \\ 7 & 0 \end{pmatrix} \begin{pmatrix} q \\ 1-q \end{pmatrix} = \begin{pmatrix} 4q+2 \\ 7q \end{pmatrix}$$

Solving $4q + 2 = 7q$ gives $q = 2/3$, hence $\vec{y} = (2/3, 1/3)$. Similarly, $\vec{x} = (2/3, 1/3)$. We can equivalently write the three Nash equilibria as the correlated equilibria

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 4/9 & 2/9 \\ 2/9 & 1/9 \end{pmatrix}$$

For the second part of the question, we characterize all correlated equilibria for the game. Suppose $z = (z_{ij})$ is a correlated equilibrium. Then the z_{ij} must satisfy the following inequalities:

$$6z_{11} + 2z_{12} \geq 7z_{11} \qquad 7z_{21} \geq 6z_{21} + 2z_{22} \qquad 6z_{11} + 2z_{21} \geq 7z_{11} \qquad 7z_{12} \geq 6z_{12} + 2z_{22}.$$

Rearranging, we find that z is a correlated equilibrium if and only if $z_{12}, z_{21} \geq z_{11}/2$ and $z_{12}, z_{21} \geq 2z_{22}$. Since the choices of $z_{11} = z_{12} = z_{21} = 1/3$ and $z_{22} = 0$ satisfy all of these inequalities,

$$\begin{pmatrix} 1/3 & 1/3 \\ 1/3 & 0 \end{pmatrix}$$

is indeed a correlated equilibrium. Finally, to find a correlated equilibrium with good payout for both players, observe that we should try to make z_{11} as large as possible (as both player I and player II benefit from a large z_{11}), and make z_{22} as small as possible. By symmetry, it suffices to consider the case where $z_{12} = z_{21}$. Since $z_{12}, z_{21} \geq z_{11}/2$, we should choose $z_{12} = z_{21} = z_{11}/2$, which gives $z_{11} = 1/2, z_{12} = z_{21} = 1/4$. The expected payoff for player I is then

$$\frac{1}{2} \cdot 6 + \frac{1}{4} \cdot 2 + \frac{1}{4} \cdot 7 + 0 \cdot 0 = 5.25.$$

Symmetrically, the payoff for player II is also 5.25, so

$$\begin{pmatrix} 1/2 & 1/4 \\ 1/4 & 0 \end{pmatrix}$$

does the job. □

Exercise 4.8. In the Game of Chicken, you should have found only three Nash equilibria. Recall that any convex combination of Nash equilibria is a correlated equilibrium. However, the converse is false in general! We can see this already in the Game of Chicken. Show that the Correlated Equilibrium

$$z = \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & 0 \end{pmatrix}$$

is not a convex combination of the Nash equilibria. Put another way, the payoffs from this Correlated Equilibrium cannot be found by randomly choosing among the Nash equilibria.

Solution. If z is a convex combination of Nash equilibria for Chicken, by the previous problem, we must have

$$z = t_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + t_2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + t_3 \begin{pmatrix} 4/9 & 2/9 \\ 2/9 & 1/9 \end{pmatrix}$$

where $t_1 + t_2 + t_3 = 0$, and $t_i \geq 0$ for all i. However, since $z_{22} = 0$, this implies that $t_3 = 0$. But if $t_3 = 0$, $z_{11} = 0$. Thus z is not the convex combination of Nash equilibria. $\square$

Exercise 4.9. Give an example of a two-person zero-sum game where there are no pure Nash equilibria. Can you give an example where all entries of the payoff matrix are different?

Solution. It is easy to see that the zero sum game with payoff matrix

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{or equivalently with bi-matrix} \quad \begin{pmatrix} (-1,1) & (1,-1) \\ (1,-1) & (-1,1) \end{pmatrix}$$

does not have any pure Nash equilibria. In particular, the unique Nash equilibrium is $((1/2, 1/2), (1/2, 1/2))$. However, it has only two distinct entries. To find a payoff matrix with distinct entries, intuitively, we should be able to gently perturb the entries of A without dramatically changing its Nash equilibria. For example, the zero sum game with payoff matrix

$$\begin{pmatrix} -1.01 & 1.01 \\ 0.99 & -0.99 \end{pmatrix}$$

does not have a pure Nash equilibrium, and all of its entries are (pairwise) distinct. $\square$

5. Homework 5

Exercise 5.1. Suppose we have a two-person zero-sum game with $(n+1) \times (n+1)$ payoff matrix A such that at least one entry of A is nonzero. Let $\vec{x}, \vec{y} \in \Delta_{n+1}$. Write $\vec{x} = (x_1, \ldots, x_n, 1 - \sum_{i=1}^{n} x_i)$, $\vec{y} = (x_{n+1}, \ldots, x_{2n}, 1 - \sum_{i=n+1}^{2n} x_i)$. Consider the function $f : \mathbb{R}^{2n} \to \mathbb{R}$ defined by $f(x_1, \ldots, x_{2n}) = \vec{x}^T A \vec{y}$. Show that the Hessian of f has at least one positive eigenvalue, and at least one negative eigenvalue. Conclude that any critical point of f is a saddle point. That is, if we find a critical point of f (as we do when we look for the value of the game), then this critical point is a saddle point of f. In this sense, the minimax value occurs at a saddle point of f.

Solution. Suppose A has entries a_{ij}. Then we can write

$$\begin{aligned} f(x_1, \ldots, x_{2n}) = \vec{x}^T A \vec{y} &= \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} x_i a_{ij} x_{j+n} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} x_i a_{ij} x_{j+n} + \sum_{j=1}^{n} (1 - x_1 - \cdots - x_n) a_{n+1,j} x_{j+n} \\ &+ \sum_{i=1}^{n} x_i a_{i,n+1} (1 - x_{n+1} - \cdots - x_{2n}) \\ &\quad + (1 - x_1 - \cdots - x_n) a_{n+1,n+1} (1 - x_{n+1} - \cdots - x_{2n}). \end{aligned}$$

Observe that f is a degree two polynomial in the variables $x_1, \ldots, x_{2n}$. Further $f(x_1, \ldots, x_{2n})$ does not contain any terms of the form cx_i^2. Therefore, we have

$$\frac{\partial^2 f}{\partial x_i^2} = 0 \quad \text{for all} \quad i = 1, \ldots, 2n.$$

Let $H = (h_{ij})$ be the Hessian matrix of f, defined by

$$h_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

By the observation above, the diagonal entries satisfy $h_{ii} = 0$ for all i, hence the trace of H (the sum of H's diagonal entries) satisfies $\text{trace}(H) = 0$. Further, the equality of mixed partial deriviatves implies that H is symmetric. Thus, by the spectral theorem, H is diagonalizable. Denote the eigenvalues of H by $\lambda_1, \ldots, \lambda_{2n}$ (possibly with repetition). Since the trace of a matrix is the sum of its eigenvalues, we have

$$\text{trace}(H) = \lambda_1 + \cdots + \lambda_{2n} = 0.$$

Finally, since A has at least one non-zero entry, H is not the zero matrix. Since H is diagonalizable, it must have at least one nonzero eigenvalue. Thus, expression $\text{trace}(H) = 0$ implies that H has at least one positive and at least one negative eigenvalue, as desired. □

Exercise 5.2. Suppose we have a two-person zero-sum game. Show that any optimal strategy is a Nash equilibrium. Then, show that any Nash equilibrium is a optimal strategy. In summary, the Nash equilibrium generalizes the notion of optimal strategy.

Solution. Note that if A is the payoff matrix for player I in a zero-sum game, then $B = -A$ is the payoff matrix for player II. First suppose $\vec{x}$ and $\vec{y}$ are optimal strategies for player I and player II respectively. Then

$$\min_{\vec{y}} \vec{x}^T A\vec{y} = \max_{\vec{x}} \min_{\vec{y}} \vec{x}^T A\vec{y}$$

and

$$\max_{\vec{x}} \vec{x}^T A\vec{y} = \min_{\vec{y}} \max_{\vec{x}} \vec{x}^T A\vec{y}.$$

Arguing as in Remark 3.30 from the notes, (using the Minimax Theorem in the middle),

$$\vec{x}^T A\vec{y} \geq \min_{\vec{y}} \vec{x}^T A\vec{y} = \max_{\vec{x}} \min_{\vec{y}} \vec{x}^T A\vec{y} = \min_{\vec{y}} \max_{\vec{x}} \vec{x}^T A\vec{y} = \max_{\vec{x}} \vec{x}^T A\vec{y} \geq \vec{x}^T A\vec{y}.$$

In particular,

$$\vec{x}^T A\vec{y} \leq \min_{\vec{y}} \vec{x}^T A\vec{y} \to \vec{x}^T B\vec{y} \geq \vec{x}^T B\vec{y} \quad \text{for all } \vec{y},$$

and

$$\vec{x}^T A\vec{y} \geq \max_{\vec{x}} \vec{x}^T A\vec{y} \to \vec{x}^T A\vec{y} \geq \vec{x}^T A\vec{y}.$$

Thus $(\vec{x}, \vec{y})$ is a Nash equilibrium.

Conversely, suppose $(\vec{x}, \vec{y})$ is a Nash equilibrium. That is, for all $\vec{x}$

$$(*) \quad \vec{x}^T A\vec{y} \geq \vec{x}^T A\vec{y} \tag{4}$$

and for all $\vec{y}$,

$$(**) \quad \vec{x}^T A\vec{y} \leq \vec{x}^T A\vec{y}. \tag{5}$$

Observe that we have

(1) $\vec{x}^T A\vec{y} \leq \min_{\vec{y}} \vec{x}^T A\vec{y}$
(2) $\leq \max_{\vec{x}} \min_{\vec{y}} \vec{x}^T A\vec{y}$
(3) $\leq \min_{\vec{y}} \max_{\vec{x}} \vec{x}^T A\vec{y}$
(4) $\leq \max_{\vec{x}} \vec{x}^T A\vec{y}$
(5) $\leq \vec{x}^T A\vec{y}$

Inequality (1) holds by (**), while (2) holds by definition of $\max_{\vec{x}}$. Inequality (3) always holds, while (4) holds by definition of $\min_{\vec{y}}$. Finally, (5) holds by (*). Since the left side of (1) is equal to the right side of (5), all terms in (1)-(5) must be equal. In particular

$$\min_{\vec{y}} \vec{x}^T A\vec{y} = \max_{\vec{x}} \min_{\vec{y}} \vec{x}^T A\vec{y} \quad \text{and} \quad \max_{\vec{x}} \vec{x}^T A\vec{y} = \min_{\vec{y}} \max_{\vec{x}} \vec{x}^T A\vec{y}.$$

Therefore, $\vec{x}$ and $\vec{y}$ are optimal strategies, as desired. □

Exercise 5.3. Show that, in any two-player general-sum game, for any $i \in \{1, 2\}$, the payoffs for player i in any Nash equilibrium exceeds the minimax value for player i. (If A is the $m \times n$ payoff matrix for player i, then the minimax value for player i is the quantity $\max_{\vec{x}\in\Delta_m} \min_{\vec{y}\in\Delta_n} \vec{x}^T A\vec{y} = \min_{\vec{y}\in\Delta_n} \max_{\vec{x}\in\Delta_m} \vec{x}^T A\vec{y}$)

Solution. For player 1, this follows immediately from inequalities (3)-(5) in the previous solution, as those inequalities depend only on (*) and the definition of $\min_{\vec{y}}$. For player 2, repeat the same argument interchanging the roles of the players using the payoff matrix B instead of A. □

Exercise 5.4. Recall that the game of Rock-Paper-Scissors is defined by the payoff matrices

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}, \quad B = A^T.$$

Then the game is symmetric. (And also, note that $A + B = 0$, so that the game is a zero-sum game.)

Show that $(1/3, 1/3, 1/3)$ is the unique Nash equilibrium. Then, show that this Nash equilibrium is **not** evolutionarily stable.

This observation leads to interesting behaviors in population dynamics. A certain type of lizard has three kinds of sub-species whose interactions resemble the Rock-Paper-Scissors game. The dynamics of the population cycle between large, dominant sub-populations of each of the three sub-species. That is, first the "Rock" lizards are a majority of the population, then the "Paper" lizards become the majority, then the "Scissors" lizards become the majority, and then the "Rock" lizards become the majority, and so on.

Solution. We first observe that Rock-Paper-Scissors (RPS) does not have pure Nash equilibria. To look for mixed equilibrium strategies, suppose $\vec{y} = (p, q, 1 - p - q)$. Then the payoffs for player I are given by

$$A\vec{y} = \begin{pmatrix} 1 - p - 2q \\ 2p + q - 1 \\ -p + q \end{pmatrix}$$

Equating $1-p-2q = -p+q$ gives $q = 1/3$, while $2p+q-1 = -p+q$ gives $p = 1/3$. Repeating this procedure for player II gives $\vec{x} = \vec{y} = (1/3, 1/3, 1/3)$, the unique Nash equilibrium.[1] Observe that for this strategy, we have $\vec{x}^T A\vec{x} = 0$.

In order to see that $\vec{x} = (1/3, 1/3, 1/3)$ is not an evolutionarily stable equilibrium, we must find a different strategy $\vec{w}$ such that

$$\vec{w}^T A\vec{x} = \vec{x}^T A\vec{x} \quad \text{and} \quad \vec{w}^T A\vec{w} > \vec{x}^T A\vec{w}.$$

Since $A\vec{x} = (0, 0, 0)$ and $\vec{x}^T A = (0 \quad 0 \quad 0)$, it suffices to find $\vec{w}$ satisfying $\vec{w}^T A\vec{w} = 0$. To this end, we observe that any pure strategy $\vec{w}$ satisfies $\vec{w}^T A\vec{w} = 0$, as the payoff to both players is 0 if they both choose the same move. For example, $\vec{w} = (1, 0, 0)$ works. □

Exercise 5.5. Let n be a positive integer. Let $v : 2^{\{1,\ldots,n\}} \to \{0, 1\}$ be a characteristic function that only takes values 0 and 1. Assume also that v is monotonic. That is, if $S, T \subseteq \{1, \ldots, n\}$ with $S \subseteq T$, then $v(S) \leq v(T)$. The Shapley-Shubik power index of each player is defined to be their Shapley value.

By monotonicity of v, we have $v(S \cup \{i\}) \geq v(S)$ for all $S \subseteq \{1, \ldots, n\}$ and for all $i \in \{1, \ldots, n\}$. Also, since v only takes values 0 and 1, we have

$$v(S \cup \{i\}) - v(S) = \begin{cases} 1 & \text{when } v(S \cup \{i\}) > v(S) \\ 0 & \text{when } v(S \cup \{i\}) = v(S) \end{cases}.$$

Consequently, we have the following simplified formula for the Shapley-Shubik power index of player $i \in \{1, \ldots, n\}$:

$$(*) \quad \phi_i(v) = \sum_{\substack{S \subseteq \{1,\ldots,n\} \\ v(S\cup\{i\})=1 \text{ and } v(S)=0}} \frac{|S|!(n - |S| - 1)!}{n!}. \tag{6}$$

Compute the Shapley-Shubik power indices for all players on the UN security council, with pre-1965 and post-1965 structure. Which structure is better for nonpermanent members?

In pre-1965 rules, the UN security council had five permanent members, and six nonpermanent members. A resolution passes only if all five permanent members want it to pass, and at least two nonpermanent members want it to pass. So, we can model this voting method, by letting $\{1, 2, \ldots, 11\}$ denote the council, and letting $\{1, 2, 3, 4, 5\}$ denote the permanent members. Then we use the characteristic function $v : 2^{\{1,\ldots,11\}} \to \{0, 1\}$ so that, for any $S \subseteq \{1, \ldots, 11\}$, $v(S) = 1$ if $\{1, 2, 3, 4, 5\} \subseteq S$ and if $|S| \geq 7$. And $v(S) = 0$ otherwise.

This voting method was called unfair, so it was restructured in 1965. After the restructuring, the council had the following form (which is still used today). The UN security council has five permanent members, and now ten nonpermanent members. A resolution passes only if all five permanent members want it to pass, and at least four nonpermanent members want it to pass. So, we can model this voting method, by letting $\{1, \ldots, 15\}$ denote the council, and letting $\{1, 2, 3, 4, 5\}$ denote the permanent members. Then we use the characteristic function $v : 2^{\{1,\ldots,15\}} \to \{0, 1\}$ so that, for any $S \subseteq \{1, \ldots, 15\}$, $v(S) = 1$ if $\{1, 2, 3, 4, 5\} \subseteq S$ and if $|S| \geq 9$. And $v(S) = 0$ otherwise.

[1] We are actually being careless here. In general, we must also check that there are no equilibrium strategies of the form $\vec{y} = (p, 1-p, 0)$, $\vec{y} = (p, 0, 1-p)$, and $\vec{y} = (0, p, 1-p)$.

Solution. We first examine the pre-1965 rules. By symmetry, all permanent members have the same power index, as do all nonpermanent members. We compute the power index of the nonpermanent members first. To this end, suppose i is a nonpermanent member, that is $i \in \{6, \ldots, 11\}$. Then notice that $S \subseteq \{1, \ldots, 11\}$ satisfies $v(S) = 0$ and $v(S \cup \{i\}) = 1$ if and only if $\{1, \ldots, 5\} \subseteq S$ and $|S| = 6$. That is, S contains all permanent members and precisely one nonpermanent member other than i. Notice that for each i there are 5 such sets S, corresponding to the 5 nonpermanent members other than i. Thus, applying (*), we find

$$\begin{aligned}
\phi_i(v) &= \sum_{\substack{S \subseteq \{1,\ldots,n\} \\ v(S\cup\{i\})=1 \text{ and } v(S)=0}} \frac{|S|!(n-|S|-1)!}{n!} \\
&= 5\frac{6!(11-6-1)!}{11!} \\
&= 5\frac{6!4!}{11!} \\
&= \frac{5 \cdot 4 \cdot 3 \cdot 2}{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7} \\
&= \frac{1}{462} \approx 0.0022.
\end{aligned}$$

Now let j be a permanent member. Instead of computing ϕ_j directly, we appeal to efficiency. That is,

$$\sum_{k \in \{1,\ldots,11\}} \phi_k(v) = v(\{1, \ldots, 11\}) = 1.$$

Since there are 5 permanent members all of whom have the same power index ($\phi_j(v)$), and 6 nonpermanent members with the same power index ($\phi_i(v)$), we therefore, have

$$5\phi_j(v) + 6\phi_i(v) = 1 \implies \phi_j(v) = \frac{1}{5}(1 - 6\phi_i(v)) = \frac{1}{5}\left(1 - \frac{6}{462}\right) = \frac{76}{385} \approx 0.197.$$

Thus, the permanent members have approximately 91 times as much power as the nonpermanent members in the pre-1965 rules.

We compute the post-1965 power indices analogously. If i is a nonpermanent member ($i \in \{6, \ldots, 15\}$), we note that if $v(S) = 0$ but $v(S \cup \{i\}) = 1$, we must have $\{1, \ldots, 5\} \subseteq S$ and $|S| = 8$ so that S contains 3 nonpermanent members other than i. Thus, by (*),

$$\phi_i(v) = \frac{\binom{9}{3}8!6!}{15!} = \frac{\binom{9}{3}8}{715} \approx 0.011.$$

If j is a nonpermanent member, then as before, efficiency gives

$$\phi_j(v) = \frac{1}{5}(1 - 10\phi_i(v)) = \frac{127}{715} \approx 0.18.$$

Thus, in the post-1965 rules, permanent members have approximately 16 times as much power as non-permanent members. □

Exercise 5.6. Let n be a positive integer. Let $v : 2^{\{1,\ldots,n\}} \to \{0, 1\}$ be a characteristic function that only takes values 0 and 1. Assume also that v is monotonic and $v(\{1, \ldots, n\}) =$

1. For each $i \in \{1, \ldots, n\}$, let B_i be the number of subsets $S \subseteq \{1, \ldots, n\}$ such that $v(S) = 0$ and $v(S \cup \{i\}) = 1$. The **Banzhaf power index** of player i is defined to be

$$\beta_i(v) = \frac{B_i}{\sum_{k=1}^n B_k}.$$

Like the Shapley-Shubik power index, the Banzhaf power index is another way to measure the relative power of each player.

Compute the Banzhaf power indices for all players for the glove market example.

Then, compute the Banzhaf power indices for all players on the UN security council, with pre-1965 and post-1965 structure. Which structure is better for nonpermanent members?

Solution. Recall that for the glove market example, we had $n = 3$ and

$$v(\{1,2\}) = v(\{1,3\}) = v(\{1,2,3\}) = 1$$

where $v(S) = 0$ for all other subsets $S \subseteq \{1,2,3\}$. Notice that $v(S) = 0$ and $v(S \cup \{1\}) = 1$ for $S = \{2\}, \{3\}, \{2,3\}$. Thus $B_1 = 3$. For $i = 2, 3$ we have $v(S) = 0$ and $v(S \cup \{i\}) = 1$ only for $S = \{1\}$. Thus $B_2 = B_3 = 1$. Thus

$$\beta_1(v) = \frac{3}{5} \quad \text{while} \quad \beta_2(v) = \beta_3(v) = \frac{1}{5}.$$

For the pre-1965 UN security council rules, suppose i is a nonpermanent member. Then as before, there are 5 sets S such that $v(S) = 0$ and $v(S \cup \{i\}) = 1$, hence $B_i = 5$. For j a permanent member $v(S) = 0$ with $v(S \cup \{j\}) = 1$ if and only if (1), S contains all other permanent members, and S contains at least 2 nonpermanent members. There are 2^6 total subsets of nonpermanent members, 1 of which contains no nonpermanent members, and 6 of which contain exactly one nonpermamament member. Thus there are $2^6 - 1 - 6 = 57$ sets of at least 2 nonpermanent members. Thus $B_j = 57$. We compute

$$\sum_k B_k = 5 \cdot 57 + 6 \cdot 5 = 315.$$

Therefore,

$$\beta_k(v) = \begin{cases} \frac{57}{315} \approx 0.18 & k \text{ is permanent member} \\ \frac{5}{315} \approx 0.016 & k \text{ is nonpermanent member} \end{cases}.$$

We compute the Banzhaf power indices for post-1965 similarly. If i is a nonpermanent member, there are $\binom{10}{3} = 84$ sets S such that $v(S) = 0$ and $v(S \cup \{i\}) = 1$, so $B_i = 84$. If j is a permanent member, then there are

$$2^{10} - \binom{10}{0} - \binom{10}{1} - \binom{10}{2} - \binom{10}{3} = 848$$

sets S such that $v(S) = 0$ and $v(S \cup \{j\}) = 1$. We compute

$$\sum_k B_k = 5 \cdot 848 + 10 \cdot 84 = 5080.$$

Thus

$$\beta_k(v) = \begin{cases} \frac{848}{5080} \approx 0.17 & k \text{ is permanent member} \\ \frac{84}{5080} \approx 0.017 & k \text{ is nonpermanent member} \end{cases}.$$

□

Exercise 5.7. Suppose we have two buyers, and $f(v) = 1$ for any $v \in [0, 1]$ in a sealed-bid second price auction. That is, V_1 and V_2 are uniformly distributed in the interval $[0, 1]$. Show that an equilibrium strategy is $\beta_1(v) = v$, $\beta_2(v) = v, \forall v \in [0, 1]$. That is, each player will bid exactly her private value.

Solution. Suppose player 2 bids $\beta_2(v) = v$ and has private valuation V_2, and player 1 has valuation v and bids b. Since we're in a second price auction, the winner pays the bid of his opponent. Note player 1 wins the auction if $b > V_2$ and loses if $b < V_2$. Thus the payoff for player 1 is

$$\begin{cases} 0 & \text{if } b < V_2, \\ v - V_2 & \text{if } b \geq V_2. \end{cases}$$

Now if $v < V_2$, player 1 wants to lose the auction, since the payoff if he won would be negative. However, if $v > V_2$, player 1 wants to win the auction, and receives the same payoff regardless of exactly how large b is in this case. The choice of b that always accomplishes these objectives is $b = v$, so $\beta_1(v) = v$.

By symmetry, if we assume $\beta_1(v) = v$, then we can conclude as above that $\beta_2(v) = v$. □

Exercise 5.8. (Muddy Children Puzzle / Blue-Eyed Islanders Puzzle). This exercise is meant to test our understanding of common knowledge.

Situation 1. There are 100 children playing in the mud. All of the children have muddy foreheads, but any single child cannot tell whether or not her own forehead is muddy. Any child can also see all of the other 99 children. The children do not communicate with each other in any way, there are no mirrors or recording devices, etc. so that no child can see her own forehead. The teacher now says, "stand up if you know your forehead is muddy." No one stands up, because no one can see her own forehead. The teacher asks again. "Knowing that no one stood up the last time, stand up now if you know your forehead is muddy." Still no one stands up. No matter how many times the teacher repeats this statement, no child stands up.

Situation 2. After Situation 1, the teacher now says, "I announce that at least one of you has a muddy forehead." The teacher then says, "stand up if you know your forehead is muddy." No one stands up. The teacher pauses then repeats, "stand up if you know your forehead is muddy." Again, no on stands up. The teacher continues making this statement. The hundredth time that she makes this statement, all the children suddenly stand up.

Explain why all of the children stand up in Situation 2, but they do not stand up in Situation 1. Pay close attention to what is common knowledge in each situation.

Solution. We first analyze situation 2. Throughout, let M denote the set of children with muddy foreheads, and C the set with clean foreheads. We argue that if k children have muddy foreheads ($|M| = k$), then after k rounds (i.e., after the teacher asks the students k times), every muddy foreheaded child $m \in M$ will stand up, while no clean $c \in C$ will stand up. We argue by induction on k. For the base case, suppose $k = 1$, i.e., $|M| = 1$. Notice that every $c \in C$ sees exactly one child with a muddy forehead, while the lone $m \in M$ sees no other muddy foreheads. When the teacher announces that at least one child has a muddy forhead, then $m \in M$ knows he is the (only) child with a muddy forhead, so he stands up. A child $c \in C$ does not stand, because she sees $m \in M$ and knows that either $|M| = 1$ (and m is the unique muddy child) or $|M| = 2$ and she is the other muddy child. Thus, she does not know whether or not she is muddy.

For the inductive step, assume that $|M| = k$. By induction, no one stood up the first $k-1$ times the teacher asked the students to stand. Notice that every $m \in M$ sees $k-1$ muddy foreheads, while each $c \in C$ sees k muddy foreheads. Therefore, each $m \in M$ knows that either there are $k-1$ muddy forheads (if m's forehead is not muddy) or k muddy foreheads (if m's forehead is muddy). Similarly, each $c \in C$ knows there are either k or $k+1$ muddy foreheads. By the indcutive hypothesis, no child stood after the $(k-1)$-st time the teach asked the students to stand. Therefore (again by the inductive hypothesis), each $m \in M$ knows that $|M| \neq k-1$. Since each $m \in M$ sees $k-1$ muddy foreheads, she must therefore know that $|M| = k$, and that she also has a muddy forehead. Thus, the next time the teacher asks she will stand.

The difference between situations 1 and 2 is most drastically seen in the base case, $k = 1$. If the teacher never tells the children that some child has a muddy forehead, then the one $m \in M$ never knows that she has a muddy forehead—for all she knows, no one has a muddy forehead. □

Exercise 5.9. There are five pirates on a ship. It is also common knowledge that every pirate prefers to maximize his amount of gold. There are 100 gold pieces to be split amongst the pirates. The game begins when the first pirate proposes how he thinks the gold should be split amongst the five pirates. All five pirates vote whether or not to accept the proposal, by a majority vote. If the proposal is accepted, the game ends. If the proposal is not accepted, the first pirate is thrown overboard, and the game continues. The second pirate now proposes how he thinks the gold should be split amongst the four remaining pirates. All four pirates vote whether or not to accept the proposal, by a majority vote (the second pirate breaks a tie). If the proposal is accepted, the game ends. If the proposal is not accepted, the second pirate is thrown overboard, and the game continues, etc. (At each stage of the game, the pirate that could be thrown overboard next can break the tie in the majority vote.) What is the largest amount of gold that the first pirate can obtain in the game?

Solution. The idea of the solution is as follows: as long as the first pirate promises at least 2 of the other other pirates more coins than they would get otherwise, they will approve and the vote passes. Label the pirates p_1, p_2, p_3, p_4, p_5, where p_1 makes the first proposal, p_2 the second, and so on. We'll work backwards from the case where p_1, p_2, and p_3 where all thrown overboard. In this case, pirate p_4 will propose that she takes all of the coins and p_5 gets nothing.

Now suppose p_1 and p_2 were thrown overboard, and p_3 makes a proposal. As long as p_3 promises to give p_5 at least one gold coin, p_5 will accept (as p_5 will get nothing if she rejects). Thus, p_3 proposes to take 99 coins, give nothing to p_4, and 1 coin to p_5. Pirates p_3 and p_5 will accept the proposal, so the proposal passes.

Now suppose only p_1 was thrown overboard. Pirate p_2 still only needs to get one additional vote (in addition to hers) for her proposal to pass. So as long as she promises p_4 one coin, p_4 will accept. So p_2 proposes that she takes 99 coins, gives 1 to p_4, who will vote in favor, while p_3 and p_5 receive nothing.

If p_1 is to have her vote accepted, she has to promise two other pirates at least as many coins as they would be guaranteed if they were to reject her proposal. Again, p_1 needs only promise p_5 a single coin for her to accept p_1's proposal. Since p_3 and p_4 are not guaranteed anything if they throw p_1 overboard, p_1 need only promise one of them a single gold coin, say p_3. Therefore, p_1 suggest that she keeps 98 coins, p_3 and p_5 keep a single coin, and p_2 and p_4 get nothing. So it seems the most that p_1 can guarantee herself is 98 gold coins! □

Exercise 5.10. Explain what a buyer in an open-bid decreasing auction knows when the current announced price is x that she did not know prior to the start of the auction. (What is common knowledge?)

Solution. Rational buyers in this auction know that the optimal strategy is to bid their private value. (This fact and anything else it implies is common knowledge.) Hence when the announced price reaches x without anyone previously bidding on the item (else the item would be auctioned off already), each buyer can conclude that the others' private values are no larger than x. □

6. Homework 6

Exercise 6.1. Suppose we have an auction with n buyers and $k < n$ is a positive integer. In a sealed-bid k-unit Vickrey auction, the top k bidders win the auction at a price equal to the $(k+1)$-st highest bid. For this auction, prove that it is a symmetric equilibrium when all buyers bid their private value.

Solution. A buyer with private value v can make a profit at most $\max(v-m, 0)$ where m is the $(k+1)$-st highest bid, and this profit is achieved when a buyer bids their private value. Moreover, this strategy dominates all other strategies. That is, each buyer with this strategy maximizes their profit, *regardless of what the other players do.* □

Exercise 6.2. In the India Premier League (IPL), cricket franchises can acquire a player by participating in the annual auction. The rules of the auction are as follows. An English auction is run until either only one bidder remains or the price reaches \$m (for example \$m could be \$750,000). In the latter case, a sealed-bid first-price auction is run with the remaining bidders. (Each of these bidders knows how many other bidders remain).

Use the Revenue Equivalence Theorem to determine equilibrium bidding strategies in an IPL cricket auction for a player with n competing franchises. Assume that the value each franchise has for this player is uniform from 0 to 1 million dollars.

Solution.

If a buyer's private value is v, and if $v < m$, then the buyer should bid up to their private value v (repeating the argument from the notes/class that this strategy dominates other strategies). Meanwhile, if $v \geq m$, then among the remaining $k \geq 2$ buyers, there is a sealed-bid first-price auction, where it is common knowledge that buyer's private values are uniform in $[m, 1]$. So *if we did* , from Theorem 7.13 in the notes, if Z is the maximum of $k-1$ i.i.d random variables that are uniform on $[0, 1-m]$, then $\mathbf{P}(Z \leq t) = t^{k-1}(1-m)^{1-k}$ and $f_Z(t) = (k-1)t^{k-2}(1-m)^{1-k}$, and the symmetric equilibrium strategy is to bid m plus

$$\frac{\int_0^{v-m} xg(x)dx}{\int_0^{v-m} g(x)dx} = \frac{\int_0^{v-m} x^{k-1}dx}{\int_0^{v-m} x^{k-2}dx} = \frac{(v-m)^k/k}{(v-m)^{k-1}/(k-1)} = (v-m)\frac{k-1}{k} = (v-m)(1-1/k).$$

That is, the bid should be

$$v(1-1/k) + m/k.$$

Note that since $k \geq 2$, this bid is smaller than v, as we would expect

□

Exercise 6.3. Prove The following Lemma from the notes: The set of functions $\{W_S\}_{S\subseteq\{1,\ldots,n\}}$ is an orthonormal basis for the space of functions from $\{-1,1\}^n \to \mathbb{R}$. (When we write $S \subseteq \{1,\ldots,n\}$, we include the empty set $\emptyset$ as a subset of $\{1,\ldots,n\}$.) (Also, for any $\vec{x} \in \{-1,1\}^n$, $W_S(\vec{x}) = \prod_{i\in S} x_i$.)

Solution. Let $V = \{f : \{-1,1\}^n \to \mathbb{R}\}$. It suffices to show that $\{W_S\}$ is an orthonormal set in V and that $|\{W_S\}| = \dim V$. First, we will show that $\dim V = 2^n$. For each $\vec{z} \in \{\pm 1\}^n$, define

$$f_{\vec{z}}(\vec{x}) = \begin{cases} 1 & \text{if } \vec{x} = \vec{z} \\ 0 & \text{if } \vec{x} \neq \vec{z} \end{cases}.$$

We claim that $B = \{f_{\vec{z}} | \vec{z} \in \{\pm 1\}^n\}$ is a basis for V. Since $|B| = 2^n$, this implies that $\dim V = 2^n$. To see that B is a basis, we must show that it is a linearly independent spanning set for V. Let $f \in V$ (that is, $f : \{\pm 1\}^n \to \mathbb{R}$). Then for each $\vec{x} \in \{\pm 1\}^n$, we can write

$$f(\vec{x}) = f(\vec{x}) \cdot 1 = f(\vec{x}) \sum_{\vec{z}\in\{\pm 1\}^n} f_{\vec{z}}(\vec{x}) = \sum_{\vec{z}\in\{\pm 1\}^n} f(\vec{z}) f_{\vec{z}}(\vec{x}).$$

Thus, every $f \in V$ can be written as a linear combination of the $f_{\vec{z}} \in B$, so B is a spanning set. To see that B is linearly independent, suppose there are coefficients $a_{\vec{z}}$ such that

$$\sum_{\vec{z}\in\{\pm 1\}^n} a_{\vec{z}} f_{\vec{z}} = 0.$$

Then for $\vec{x} \in \{\pm 1\}^n$, we have

$$0 = \sum_{\vec{z}\in\{\pm 1\}^n} a_{\vec{z}} f_{\vec{z}}(\vec{x}) = a_{\vec{x}} f_{\vec{x}}(\vec{x}) = a_{\vec{x}}.$$

Thus, $a_{\vec{z}} = 0$ for all $\vec{z}$, implying that B is linearly indpendent. Therefore, $\dim V = 2^n$, as desired. Since there are 2^n subsets $S \subseteq \{1,\ldots,n\}$, in order to show $\{W_S\}$ is an orthonormal basis, it suffices to show $\{W_S\}$ is an orthonormal set.

To show $\{W_S\}$ is an orthonormal set, we first compute for each $S \subseteq \{1,\ldots,n\}$

$$\begin{aligned}
\langle W_S, W_S\rangle &= 2^{-n} \sum_{\vec{x}\in\{\pm 1\}^n} W_S(\vec{x}) W_S(\vec{x}) \\
&= 2^{-n} \sum_{\vec{x}\in\{\pm 1\}^n} \left(\prod_{i\in S} x_i\right)\left(\prod_{i\in S} x_i\right) \\
&= 2^{-n} \sum_{\vec{x}\in\{\pm 1\}^n} \left(\prod_{i\in S} x_i^2\right) \\
&= 2^{-n} \sum_{\vec{x}\in\{\pm 1\}^n} 1 \quad (x_i = 1 \text{ or } -1) \\
&= 2^{-n}(2^n) = 1.
\end{aligned}$$

Now suppose $S \neq T$. In particular, assume that there exists $k \in S$ with $k \notin T$. Then we compute

$$\begin{aligned}
\langle W_S, W_T \rangle &= 2^{-n} \sum_{\vec{x} \in \{\pm 1\}^n} W_S(\vec{x}) W_T(\vec{x}) \\
&= 2^{-n} \sum_{x_1, \ldots, x_k \in \{\pm 1\}} \left(\prod_{i \in S, i \neq k} x_i \right) \left(\prod_{j \in T} x_j \right) \\
&= 2^{-n} \sum_{\substack{x_i \in \{\pm 1\} \\ i \neq k}} \left(\prod_{i \in S, i \neq k} x_i \right) \left(\prod_{j \in T} x_j \right) \left(\sum_{x_k \in \{\pm 1\}} x_k \right) \\
&= \sum_{\substack{x_i \in \{\pm 1\} \\ i \neq k}} \left(\prod_{i \in S, i \neq k} x_i \right) \left(\prod_{j \in T} x_j \right) (0) \\
&= 0.
\end{aligned}$$

Thus, $\{W_S\}$ is an orthonormal set with 2^n elements, hence an orthonormal basis for V. □

Exercise 6.4. Let $f : \{-1, 1\}^2 \to \{-1, 1\}$ such that $f(x) = 1$ for all $x \in \{-1, 1\}^2$. Compute $\hat{f}(S)$ for all $S \subseteq \{1, 2\}$.

Let $f : \{-1, 1\}^3 \to \{-1, 1\}$ such that $f(x_1, x_2, x_3) = \text{sign}(x_1 + x_2 + x_3)$ for all $(x_1, x_2, x_3) \in \{-1, 1\}^3$. Compute $\hat{f}(S)$ for all $S \subseteq \{1, 2, 3\}$. The function f is called a **majority function**.

Solution. First consider $f : \{\pm 1\}^2 \to \{\pm 1\}$ given by $f(x) = 1$. Observe that $W_\emptyset(x) = 1$ for all x as well, hence $f = W_\emptyset$. By the previous exercise, we have

$$\hat{f}(S) = \langle f, W_S \rangle = \begin{cases} 1 & \text{if } S = \emptyset \\ 0 & \text{if } S \neq \emptyset. \end{cases}$$

Now consider f the majority function as defined above. Specifically, f has values

$$\begin{gathered}
f(1,1,1) = f(-1,1,1) = f(1,-1,1) = f(1,1,-1) = 1 \\
f(-1,-1,1) = f(-1,1,-1) = f(1,-1,-1) = f(-1,-1,-1) = -1.
\end{gathered}$$

We compute

$$\begin{aligned}
\hat{f}(\emptyset) = \langle f, W_\emptyset \rangle &= 2^{-n} \sum_{\vec{x} \in \{\pm 1\}^n} f(\vec{x}) W_\emptyset(\vec{x}) \\
&= 2^{-n} \sum_{\vec{x} \in \{\pm 1\}^n} f(\vec{x}) \cdot 1 \\
&= 2^{-3}(1 + 1 + 1 + 1 - 1 - 1 - 1 - 1) \\
&= 0.
\end{aligned}$$

For $S=\{1\}$, we have $W_S(x_1,x_2,x_3)=x_1$. Therefore

$$\hat{f}(\{1\}) = \langle f, W_1\rangle = 2^{-n}\sum_{\vec{x}\in\{\pm 1\}^n} f(\vec{x})W_1(\vec{x})$$

$$= 2^{-3}\sum_{\vec{x}\in\{\pm 1\}^n} f(x_1,x_2,x_3)x_1$$

$$= \frac{1}{8}((1)(1)+(1)(-1)+(1)(1)+(1)(1)+(-1)(-1)+(-1)(-1)+(-1)(1)+(-1)(-1))$$

$$= \frac{1}{2}.$$

Similarly, we find that $\hat{f}(S)=1/2$ for $|S|=1$. When $|S|=2$, we find $\hat{f}(S)=0$, and when $S=\{1,2,3\}$, $\hat{f}(S)=-1/2$. Thus

$$\hat{f}(S) = \begin{cases} 0 & \text{if } |S|=0,2 \\ 1/2 & \text{if } |S|=1 \\ -1/2 & \text{if } |S|=3. \end{cases}$$

□

Exercise 6.5. Let $f\colon\{-1,1\}^3\to\{-1,1\}$ such that $f(x_1,x_2,x_3)=\mathrm{sign}(x_1+x_2+x_3)$ for all $(x_1,x_2,x_3)\in\{-1,1\}^3$. In the previous homework, we computed $\hat{f}(S)$ for all $S\subseteq\{1,2,3\}$. The function f is called a **majority function**. Compute the noise stability of f, for any $\rho\in(-1,1)$.

Let n be a positive odd integer. The majority function for n voters can be written as $f(x_1,\ldots,x_n)=\mathrm{sign}(x_1+\cdots+x_n)$, where $x_1,\ldots,x_n\in\{-1,1\}$ and $f\colon\{-1,1\}^n\to\{-1,1\}$. In the limit as $n\to\infty$, the noise stability of the majority function approaches a limiting value. (We implicitly used this fact in stating the Majority is Stablest Theorem.) You will compute this limiting value A in the following way. We have $A=4B-1$, where B is defined below. Let z_1,z_2 be vectors of unit length in $\mathbb{R}^2$. Let $\rho\in(-1,1)$. Let $\cdot$ denote the standard inner product of vectors in $\mathbb{R}^2$. Assume that $z_1\cdot z_2=\rho$. Let $C\subseteq\mathbb{R}^2$ be the set such that

$$C=\{x\in\mathbb{R}^2 : x\cdot z_1\geq 0 \text{ and } x\cdot z_2\geq 0\}.$$

Then

$$B=\int_C \frac{e^{-(x^2+y^2)/2}}{2\pi}dxdy.$$

Compute the value of A. (You should get a relatively simple quantity involving an inverse trigonometric function.)

Solution. By the computation in the previous exercise, we compute the noise stability of the majority function f with $n=3$ by

$$\langle f, T_\rho f\rangle = \sum_{S\subseteq\{1,2,3\}} \rho^{|S|}|\hat{f}(S)|^2$$

$$= 3\rho\left(\frac{1}{2}\right)^2 + \rho^3\left(\frac{-1}{2}\right)^2$$

$$= \frac{3}{4}\rho + \frac{1}{4}\rho^3.$$

To compute the integral

$$B = \int_C \frac{e^{-(x^2+y^2)/2}}{2\pi} dxdy,$$

first observe that that the region C is an infinte "wedge" centered at the origin consisting of points $\vec{x}$ satisfying $\vec{x} \cdot \vec{z}_1 \geq 0$ and $\vec{x} \cdot \vec{z}_2 \geq 0$. The angle of the wedge is $\pi - \psi$, where ψ is the angle between $\vec{z}_1$ and $\vec{z}_2$. We compute ψ by

$$\rho = \vec{z}_1 \cdot \vec{z}_2 = ||\vec{z}_1|| ||\vec{z}_2|| \cos \psi \implies \psi = \cos^{-1}(\rho)$$

because $||\vec{z}_1|| = ||\vec{z}_2|| = 1$. Thus, the angle of the wedge is $\Theta = \pi - \psi = \pi - \cos^{-1}(\rho) = \cos^{-1}(-\rho)$. The region C can be expressed in polar coordinates as

$$C = \{(r, \theta) : 0 \leq r < \infty, \theta_1 \leq \theta \leq \theta_2\} \quad \text{where } \theta_2 - \theta_1 = \Theta.$$

Notice that by symmetry, the integral for B depends only on Θ, not on the actual values of θ_1 and θ_2. Therefore, a change of variables to polar coordinates gives

$$\begin{aligned} B &= \int_C \frac{e^{-(x^2+y^2)/2}}{2\pi} dxdy \\ &= \frac{1}{2\pi} \int_0^\infty \int_{\theta_1}^{\theta_2} re^{-r^2/2} d\theta dr \\ &= \frac{\Theta}{2\pi} \int_0^\infty re^{-r^2/2} dr \\ &= \frac{\Theta}{2\pi} \int_0^\infty e^{-u} du \quad (u = r^2/2) \\ &= \frac{\Theta}{2\pi}. \end{aligned}$$

Since $\Theta = \cos^{-1}(-\rho)$, we find that

$$A = 4B - 1 = \frac{2\cos^{-1}(-\rho)}{\pi} - 1.$$

□

Exercise 6.6. Let f denote the majority function for n voters. In class, we showed that $I_i(f) \approx 1/\sqrt{n}$ for all $i \in \{1, \ldots, n\}$. Explain why we can interpret this calculation as saying: your influence in a majority election is a lot more than $1/n$, so you should vote. On the other hand, give reasons why the influence calculation may not accurately reflect your actual influence in a majority election. (If you are thinking of elections in the US, feel free to consider or ignore the electoral college system.)

Solution. We can interpret $I_i(f)$ as the probability that x_i determines the outcome of an election assuming that all x_j for $j \neq i$ are assigned randomily. The fact that for a majority vote, $I_i(f) \approx 1/\sqrt{n}$ means that the likelyhood that any particular person's vote changes the outcome of an election is approximately $1/\sqrt{n}$. This fact is perhaps counterintuitive, as all of the n votes count equally–one might naively expect, therefore, that the probability of one particular vote changing the outcome of an election to be $1/n$. Since your influence as a voter is relatively large, you should vote!

The problem with this intuition (and the definition of $I_i(f)$ from a practical standpoint) is that it assumes that all other votes are cast randomly. In particular, it assumes that every voter $j \neq i$ is equally likely to vote for both candidates. In reality, this would be very unusual. If, on the other hand, voters are even slightly more likely to vote for candidate A than candidate B, then an individual's influence on the outcome of the election is much smaller. $\square$

Exercise 6.7. Let n be a positive integer. Let $f, g : \{-1,1\}^n \to \{-1,1\}$. Let $a_0, \ldots, a_n, b_0, \ldots, b_n \in \mathbb{R}$. Let $\vec{x} = (x_1, \ldots, x_n) \in \{-1,1\}^n$. For any $\vec{x} \in \{-1,1\}^n$, define

$$L_f(\vec{x}) = a_0 + \sum_{i=1}^n a_i x_i, \quad L_g(\vec{x}) = b_0 + \sum_{i=1}^n b_i x_i.$$

Assume that $L_f(\vec{x}) \neq 0$ and $L_g(\vec{x}) \neq 0$ for all $\vec{x} \in \{-1,1\}^n$. Assume also that $f(\vec{x}) = \mathrm{sign}(L_f(\vec{x}))$ and $g(\vec{x}) = \mathrm{sign}(L_g(\vec{x}))$ for all $\vec{x} \in \{-1,1\}^n$. Assume that $\hat{f}(S) = \hat{g}(S)$ for all $S \subseteq \{1, \ldots, n\}$ with $|S| \leq 1$. Prove that $f = g$. (Hint: what does the Plancherel Theorem say about $\langle f, L_f \rangle$? How does this quantity compare with $\langle g, L_f \rangle$ and $\langle g, L_g \rangle$? Also, note that $f(\vec{x})L_f(\vec{x}) = |L_f(\vec{x})| \geq g(\vec{x})L_f(\vec{x})$ for any $\vec{x} \in \{-1,1\}^n$.)

Solution. First, observe that $W_\emptyset(\vec{x}) = 1$, while $W_i(\vec{x}) = x_i$ for all $i \in \{1, \ldots, n\}$. Therefore, we have

$$L_f(\vec{x}) = a_0 + \sum_{i=1}^n a_i x_i = a_0 W_\emptyset(\vec{x}) + \sum_{i=1}^n a_i W_i(\vec{x}).$$

Therefore,

$$\hat{L_f}(S) = \begin{cases} a_0 & \text{if } S = \emptyset \\ a_i & \text{if } S = \{i\} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, applying Plancherel's theorem, we compute

$$\begin{aligned} \langle f, L_f \rangle &= \sum_{S \subseteq \{1,\ldots,n\}} \hat{f}(S)\hat{L_f}(S) \\ &= \sum_{|S| \leq 1} \hat{f}(S)\hat{L_f}(S) \\ &= \sum_{|S| \leq 1} \hat{g}(S)\hat{L_f}(S) \\ &= \langle g, L_f \rangle. \end{aligned}$$

The third equality holds because of the hypthesis that $\hat{f}(S) = \hat{g}(S)$ for all S with $|S| \leq 1$. On the other hand, we compute

$$\langle f, L_f \rangle = 2^{-n} \sum_{\vec{x} \in \{\pm 1\}^n} f(\vec{x})L_f(\vec{x}) = 2^{-n} \sum_{\vec{x} \in \{\pm 1\}^n} |L_f(\vec{x})|$$

and similarly,

$$\langle g, L_f \rangle = 2^{-n} \sum_{\vec{x} \in \{\pm 1\}^n} g(\vec{x})L_f(\vec{x}).$$

Since $g(\vec{x}) \in \{\pm 1\}$, we have $g(\vec{x})L_f(\vec{x}) \leq |L_f(\vec{x})|$ for all $\vec{x}$ with equality if and only if $g(\vec{x}) = \text{sign}(L_f(\vec{x})) = f(\vec{x})$. Therefore, since

$$2^{-n} \sum_{\vec{x}\in\{\pm 1\}^n} g(\vec{x})L_f(\vec{x}) = \langle g, L_f\rangle = \langle f, L_f\rangle = 2^{-n} \sum_{\vec{x}\in\{\pm 1\}^n} |L_f(\vec{x})|,$$

we must have $f(\vec{x}) = g(\vec{x})$ for all $\vec{x}$, as desired. $\square$

Exercise 6.8. Let n be a positive integer. Show that there is a one-to-one correspondence (or a bijection) between the set of functions f where $f : \{-1, 1\}^n \to \mathbb{R}$, and the set of functions g where $g : 2^{\{1,2,\ldots,n\}} \to \mathbb{R}$. For example, you could identify a subset $S \subseteq \{1, \ldots, n\}$ with the element $\vec{x} = (x_1, \ldots, x_n) \in \{-1, 1\}^n$ where, for all $i \in \{1, \ldots, n\}$, we have $x_i = 1$ if $i \in S$, and $x_i = -1$ if $i \notin S$. Let $i, j \in \{1, \ldots, n\}$ and let $\vec{x} \in \{-1, 1\}^n$. Let $S(\vec{x}) = \{j \in \{1, \ldots, n\} : x_j = 1\}$. Using this one-to-one correspondence, show that the i^{th} Shapley value of $f : \{-1, 1\}^n \to \{-1, 1\}$ can be written as

$$\phi_i(f) = \sum_{\substack{\vec{x}\in\{-1,1\}^n \\ x_i=-1}} \frac{|S(\vec{x})|!(n - |S(\vec{x})| - 1)!}{n!} \left(f(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) - f(\vec{x})\right).$$

So, $\phi_i(f)$ is similar to, but distinct from, $I_i(f)$. On the other hand, the i^{th} Banzhaf power index is essentially identical to $I_i(f)$. That is, if we define

$$B_i(f) = \sum_{\vec{x}\in\{-1,1\}^n} \frac{f(x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n) - f(\vec{x})}{2},$$

then $B_i(f)/\sum_{j=1}^n B_j(f)$ is the i^{th} Banzhaf power index of f.

Solution. Consider the function $\psi : \{\pm 1\}^n \to 2^{\{1,2,\ldots,n\}}$ given by

$$\psi(\vec{x}) = S \quad \text{where } S = \{i | x_i = 1\}.$$

To see that ψ is a bijection, observe that the inverse of ψ is given by $\psi^{-1} : 2^{\{1,\ldots,n\}} \to \{\pm 1\}^n$ where

$$\psi^{-1}(S) = \vec{x} \text{ where } x_i = \begin{cases} 1 & \text{if } i \in S \\ -1 & \text{if } i \notin S \end{cases}.$$

Now for any $g : 2^{\{1,\ldots,n\}} \to \mathbb{R}$, define the function $f : \{\pm 1\}^n \to \mathbb{R}$ given by $f(\vec{x}) = g(\psi(\vec{x}))$. Similarly, given a function $f : \{\pm 1\}^n \to \mathbb{R}$ we can define $g : 2^{\{1,\ldots,n\}} \to \mathbb{R}$ by $g(S) = f(\psi^{-1}(S))$. Since ψ is a bijection, it is clear that the correspondence $f \leftrightarrow g$ is also a bijection.

Recall that given a characteristic function $g : 2^{\{1,\ldots,n\}} \to \mathbb{R}$, we can compute the i-th Shapley value of g to be

$$\phi_i(g) = \sum_{S : i\notin S} \frac{|S|!(n - |S| - 1)!}{n!}(v(S \cup \{i\}) - v(S)).$$

Given $f : \{\pm 1\}^n \to \mathbb{R}$, we can define the Shapley values of f by $\phi_i(f) = \phi_i(g)$ where $f = g \circ \psi$. Therefore we obtain

$$\begin{aligned}\phi_i(f) &= \phi_i(g)\\ &= \sum_{S: i \notin S} \frac{|S|!(n-|S|-1)!}{n!}(v(S \cup \{i\}) - v(S))\\ &= \sum_{\substack{\vec{x} \in \{\pm 1\}^n \\ x_i = -1}} \frac{|\psi(\vec{x})|!(n-|\psi(\vec{x})|-1)!}{n!}(f(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) - f(\vec{x})).\end{aligned}$$

This gives the desired result.

Similarly, using the bijection on the formula for Banzhaf power indices yields the final expression in the exercise. □

Exercise 6.9. Let $f : \{-1,1\}^n \to \{-1,1\}$. Assume that $\hat{f}(S) = 0$ whenever $S \subseteq \{1, \ldots, n\}$ and $|S| \neq 1$. Show that there exists $i \in \{1, \ldots, n\}$ such that $f(\vec{x}) = f(x_1, \ldots, x_n) = x_i$ for all $\vec{x} \in \{-1,1\}^n$, or $f(\vec{x}) = -x_i$ for all $\vec{x} \in \{-1,1\}^n$.

Solution. Since $\hat{f}(S) = 0$ for all S with $|S| > 1$, we have

$$f = \sum_{S \subseteq \{1, \ldots, n\}} \langle f, W_S \rangle W_S = \sum_{S: |S| \leq 1} \langle f, W_S \rangle W_S.$$

Since $W_\emptyset(\vec{x}) = 1$ for all $\vec{x}$, and $W_i(\vec{x}) = x_i$ for all $i = 1, \ldots, n$. Therefore, we can write

$$f(\vec{x}) = a_0 + \sum_{i=1}^n a_i x_i \quad \text{where } a_0 = \langle f, W_\emptyset \rangle \text{ and } a_i = \langle f, W_i \rangle.$$

Therefore, we must show that there is precisely one $j \in \{0, 1, \ldots, n\}$ such that $a_j = \pm 1$, while $a_i = 0$ for all $i \neq j$. We first show by induction on n that each a_i satisfies $a_i \in \{1, 0, -1\}$. It is clear that for $n = 0$, we must have $a_0 = \pm 1$. For the inductive step, write $\vec{x} \in \{\pm 1\}^n$ by $(\vec{x'}, x_n)$ where $\vec{x'} \in \{\pm 1\}^{n-1}$ and $x_n \in \{\pm 1\}$. Then for any fixed $\vec{x'}$, we must have

$$f(\vec{x'}, 1), f(\vec{x'}, -1) \in \{\pm 1\} \quad \text{so that} \quad f(\vec{x'}, 1) - f(\vec{x'}, -1) = \begin{cases} 0 & \text{if } f(\vec{x'}, 1) = f(\vec{x'}, -1) \\ \pm 2 & \text{if } f(\vec{x'}, 1) \neq f(\vec{x'}, -1). \end{cases}$$

Therefore, $a_i \in \{1, 0, -1\}$ for all i. Now consider $\vec{x} \in \{\pm 1\}^n$ given by $x_i = \text{sign}(a_i)$ if $a_i \neq 0$, and x_i arbitrary otherwise. Then

$$f(\vec{x}) = a_0 + \sum_{i=1}^n a_i x_i.$$

Since this value must be -1 or 1, with all of the a_i's equal to 1, 0, or -1, there can clearly be at most one a_k with $a_k \neq 0$. This gives the desired result. □

7. Homework 7

Exercise 7.1. Show that, in a Condorcet election with three candidates, if we use a majority vote to compare each pair of candidates, then as the number of voters goes to infinity, the probability that a Condorcet winner occurs approaches

$$\frac{3}{4} + \frac{3}{2\pi}\sin^{-1}(1/3).$$

(Assume that each voter ranks their candidates uniformly at random and independently of all other voters.)

Solution. From (the proof of) Arrow's Impossibility Theorem in the notes, the probability of a Condorcet winner occurring is

$$\frac{3}{4}(1 - \langle f, T_{-1/3} f\rangle),$$

where $f\colon \{-1,1\}^n \to \{-1,1\}$ is the voting method defining the Condorcet election (which is the majority function). From a previous exercise (or the notes), the noise stability of majority (as the number of voters goes to infinity) is $\langle f, T_\rho f\rangle = \frac{2}{\pi}\sin^{-1}(\rho)$ for any $-1 < \rho < 1$. So, plugging in $\rho = -1/3$, the probability of a paradox is

$$\frac{3}{4}(1 - \frac{2}{\pi}\sin^{-1}(-1/3)) = \frac{3}{4}(1 + \frac{2}{\pi}\sin^{-1}(1/3)).$$

□

Exercise 7.2. Create an algorithm that finds an evolutionarily stable strategy in a two-person general sum game (if the strategy exists). Find a reasonable bound on the run time of this algorithm.

[See next exercise]

Exercise 7.3. Find all Nash equilibria of a two-player symmetric game with the following payoffs

$$\begin{pmatrix} 4 & 3 & 2 & 5 & 6 \\ 3 & 1 & 8 & 9 & 1 \\ 7 & 0 & 7 & 0 & 7 \\ 1 & 3 & 3 & 2 & 1 \\ 0 & 1 & 2 & 9 & 1 \end{pmatrix}$$

(Hint: maybe you should use a computer program.)

Find all evolutionary stable strategies.

(Hint: maybe you should use a computer program.)

The program below finds all Nash equilibria of the game.

```
import numpy as np
from itertools import combinations

def all_nonempty_subsets(n):
    """Generate all nonempty subsets of {0,...,n-1}."""
    return [list(c) for r in range(1, n+1) for c in combinations(range(n), r)]

def solve_support_pair_general(A, B, I, J, tol=1e-8):
    """
    Try to find a Nash equilibrium with supports I (row) and J (col).
    Returns (x, y) if successful, else None.
```

```
A, B are the m by n payoff matrices, specified as numpy arrays
"""
m, n = A.shape
size_I = len(I)
size_J = len(J)
num_vars = size_I + size_J  # x_i (i in I) + y_j (j in J)

x_idx = {i: k for k, i in enumerate(I)}
y_idx = {j: k + size_I for k, j in enumerate(J)}

C = []

# For each i in I, we want: (A y)_i = constant over support
# we write this as: (Ay)_i - (Ay)_i0  = 0, for all i in I except i0
#  that is  A_mod y =0 where A_mod_ij = A_ij - A_i0j for all i in I
#    also since y_j=0 for all j not in J, we ignore indices outside of J
i0 = I[0]
for i in I[1:]:
    row = np.zeros(num_vars)
    for j in J:
        row[y_idx[j]] += A[i, j] - A[i0, j]
    C.append(row.tolist())

# For j in J: (x^T B)_j = constant over support
# we write this as: (x^T B)_j - (x^T B)_j0  = 0, for all j in J except j0
#  that is  x^ T B_mod =0 where B_mod_ij = B_ij - B_ij0 for all j in J
#  that is, B_mod^T x =0

j0 = J[0]
for j in J[1:]:
    col = np.zeros(num_vars)
    for i in I:
        col[x_idx[i]] += B[i, j] - B[i, j0]
    C.append(col.tolist())

# combine equations in block form, to get [A, 0; 0, B] [y, x]^T =0
# but then also add constraint on sum being 1 to get
# [A, 0; 0, B^T; 1...1 0...0; 0...0 1...1] [y, x]^T =0

C.append([1]*size_I + [0]*size_J)
C.append([0]*size_I + [1]*size_J)

#  Finally, solve Cz =0.  C is |I|+|J|+2 X |I|+|J|+2, b is
```

```
    C = np.array(C)
    b = np.array([0] * num_vars)
    b[-2] = 1
    b[-1] = 1
    z = np.linalg.solve(C, b)

    x = np.zeros(m)
    y = np.zeros(n)
    for i in I:
        x[i] = z[x_idx[i]]
    for j in J:
        y[j] = z[y_idx[j]]

    # Check that all entries of x, y are nonnegative, and maximizing property holds:
    #  (Ay)_i <= x^T Ay for all indices i, and (Ay)_i <= x^T Ay
    player_1_payoff = x.T @ A @ y
    player_2_payoff = x.T @ B @ y
    if  all((x[i] > -tol) for i in range(m)) and \
        all((y[j] > -tol) for j in range(n)) and \
        all((A[i] @ y <= player_1_payoff + tol) for i in range(m)) and \
        all((x @ B[:, j] <= player_2_payoff + tol) for j in range(n)):
        return x, y
    return None

def find_nash_equilibrium_general(A, B, tol=1e-14):
    """Find all Nash equilibria in a 2-player general-sum game."""
    m, n = A.shape
    result=[]
    for I in all_nonempty_subsets(m):
        for J in all_nonempty_subsets(n):
            try:
                new_result = solve_support_pair_general(A, B, I, J, tol)
                result.append(new_result)
            except:
                -

    if result:
        return result
    else:
        return None


def is_ESS(A, B, x, y, tol=1e-8):
    """Checks if a Nash Equilibria is an Evoluationarilyi Stable Strategy."""
```

```
    m, n = A.shape
    if m != n:
        return False

    # first, check if x,y is symmetric
    if  any((abs(x[i] - y[i]) > tol) for i in range(len(A))):
        return False

    # then, check for invasive populations
    for i in range(m):
        w = np.zeros(m)
        w[i] = 1
        if abs(w.T @ A @ x - w.T @ A @ x) < tol and (w.T @ A @ w >= x.T @ A @ w - tol):
            return False

    return True

if __name__ == "__main__":
    A = np.array(
        [[4, 3, 2, 5, 6],
        [3, 1, 8, 9, 1],
        [7, 0, 7, 0, 7],
        [1, 3, 3, 2, 1],
        [0, 1, 2, 9, 1]]
    )
    B = A.T

    result = find_nash_equilibrium_general(A, B)
    num_equilibria = 0
    if result:
        print(" Nash Equilibrium Found:")
        for equilibria in result:
            if equilibria:
                x, y = equilibria
                print("  Player 1 strategy: x=", x, ". Player 2 strategy: y=", y, "Is ES
                num_equilibria += 1
        print("Total Number of Equilibria (with duplicates):", num_equilibria)
    else:
        print("No equilibria found.")
```

The output of this program was:

```
Nash Equilibrium Found:
  Player 1 strategy: x= [0. 1. 0. 0. 0.] . Player 2 strategy: y= [0. 0. 0. 1. 0.] Is ESS
  Player 1 strategy: x= [0. 0. 0. 1. 0.] . Player 2 strategy: y= [0. 1. 0. 0. 0.] Is ESS
  Player 1 strategy: x= [0.69230769 0.         0.         0.30769231 0.        ] . Playe
  Player 1 strategy: x= [-0.  0.  0.  1.  0.] . Player 2 strategy: y= [ 0.  1.  0.  0. -
```

```
  Player 1 strategy: x= [0. 1. 0. 0. 0.] . Player 2 strategy: y= [0.69230769 0.
  Player 1 strategy: x= [ 0.  1.  0.  0. -0.] . Player 2 strategy: y= [-0.  0.  0.  1.
  Player 1 strategy: x= [0.2   0.575 0.225 0.    0.   ] . Player 2 strategy: y= [0.2   0
  Player 1 strategy: x= [0.08 0.68 0.24 0.   0.  ] . Player 2 strategy: y= [0.57894737 0
  Player 1 strategy: x= [0.57894737 0.18421053 0.         0.23684211 0.        ] . Playe
Total Number of Equilibria (with duplicates): 9
```

So, after deleting some duplicates, we obtained 7 Nash equilibria, none of which are ESS.

```
  Player 1 strategy: x= [0. 1. 0. 0. 0.] . Player 2 strategy: y= [0. 0. 0. 1. 0.] Is ESS
  Player 1 strategy: x= [0. 0. 0. 1. 0.] . Player 2 strategy: y= [0. 1. 0. 0. 0.] Is ESS
  Player 1 strategy: x= [0.69230769 0.         0.         0.30769231 0.        ] . Playe
  Player 1 strategy: x= [0. 1. 0. 0. 0.] . Player 2 strategy: y= [0.69230769 0.
  Player 1 strategy: x= [0.2   0.575 0.225 0.    0.   ] . Player 2 strategy: y= [0.2   0
  Player 1 strategy: x= [0.08 0.68 0.24 0.   0.  ] . Player 2 strategy: y= [0.57894737 0
  Player 1 strategy: x= [0.57894737 0.18421053 0.         0.23684211 0.        ] . Playe
```

THE EXERCISES BELOW WERE OPTIONAL.

Exercise 7.4 (Optional)**.** Prove a Hoeffding inequality for random variables $X_1, \ldots, X_n$ that are 1-sub-Gaussian.

A real-valued random variable X is called k-sub-Gaussian if $\mathbf{E}|X| < \infty$ and

$$\mathbf{E}e^{[X-\mathbf{E}X]t} \leq e^{k^2t^2/2}, \qquad \forall\, t \in \mathbb{R}.$$

Solution. This is a repetition of the argument from the notes. □

Exercise 7.5 (Optional)**.** Write a computer program that implements the bandit algorithms we discussed in class. (Explore-then-Commit, Successive Elimination, and UCB) Consider e.g. some rewards with different expected values, and plot the regret over time. Compare your findings with our theoretical regret bounds.

Try also rewards that are Gaussian random variables with different means. How do your regret bounds behave?

Try also rewards that are Poisson with different means. How do your regret bounds behave?

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