Please provide complete and well-written solutions to the following exercises.

Due April 17, 1159PM PST, to be uploaded as a single PDF document to Brightspace.

Homework 6

Exercise 1. Suppose we have an auction with n buyers and $k < n$ is a positive integer. In a sealed-bid k-unit Vickrey auction, the top k bidders with the auction at a price equal to the $(k+1)$-st highest bid. For this auction, prove that it is a symmetric equilibrium when all buyers bid their private value.

Exercise 2. In the India Premier League (IPL), cricket franchises can acquire a player by participating in the annual auction. The rules of the auction are as follows. An English auction is run until either only one bidder remains or the price reaches \$m (for example \$m could be \$750,000). In the latter case, a sealed-bid first-price auction is run with the remaining bidders. (Each of these bidders knows how many other bidders remain).

Use the Revenue Equivalence Theorem to determine equilibrium bidding strategies in an IPL cricket auction for a player with n competing franchises. Assume that the value each franchise has for this player is uniform from 0 to 1 million dollars.

Exercise 3. Prove the following Lemma from the notes: The set of functions $\{W_S\}_{S\subseteq\{1,\ldots,n\}}$ is an orthonormal basis for the space of functions from $\{-1,1\}^n \to \mathbf{R}$, with respect to the inner product defined in the notes. (When we write $S \subseteq \{1,\ldots,n\}$, we include the empty set $\emptyset$ as a subset of $\{1,\ldots,n\}$.) (Also, for any $x \in \{-1,1\}^n$, $W_S(x) = \prod_{i\in S} x_i$.)

Exercise 4. Let $f\colon \{-1,1\}^2 \to \{-1,1\}$ such that $f(x) = 1$ for all $x \in \{-1,1\}^2$. Compute $\widehat{f}(S)$ for all $S \subseteq \{1,2\}$.

Let $f\colon \{-1,1\}^3 \to \{-1,1\}$ such that $f(x_1,x_2,x_3) = \mathrm{sign}(x_1+x_2+x_3)$ for all $(x_1,x_2,x_3) \in \{-1,1\}^3$. Compute $\widehat{f}(S)$ for all $S \subseteq \{1,2,3\}$. The function f is called a **majority function**.

Exercise 5. Let $f\colon \{-1,1\}^3 \to \{-1,1\}$ such that $f(x_1,x_2,x_3) = \mathrm{sign}(x_1+x_2+x_3)$ for all $(x_1,x_2,x_3) \in \{-1,1\}^3$. In the previous exercise, we computed $\widehat{f}(S)$ for all $S \subseteq \{1,2,3\}$. The function f is called a **majority function**. Compute the noise stability of f, for any $\rho \in (-1,1)$.

Let n be a positive odd integer. The majority function for n voters can be written as $f(x_1,\ldots,x_n) = \mathrm{sign}(x_1+\cdots+x_n)$, where $x_1,\ldots,x_n \in \{-1,1\}$ and $f\colon \{-1,1\}^n \to \{-1,1\}$. In the limit as $n \to \infty$, the noise stability of the majority function approaches a limiting value. (We implicitly used this fact in stating the Majority is Stablest Theorem.) You will compute this limiting value A in the following way. We have $A = 4B - 1$, where B is defined below.

Let z_1, z_2 be vectors of unit length in $\mathbf{R}^2$. Let $\rho \in (-1,1)$. Let $\cdot$ denote the standard inner product of vectors in $\mathbf{R}^2$. Assume that $z_1 \cdot z_2 = \rho$. Let $C \subseteq \mathbf{R}^2$ be the set such that

$$C = \{(x,y) \in \mathbf{R}^2 \colon (x,y)\cdot z_1 \geq 0 \text{ and } (x,y)\cdot z_2 \geq 0\}.$$

Then

$$B = \iint_C e^{-(x^2+y^2)/2} \frac{dxdy}{2\pi}.$$

Compute the value of A. (You should get a relatively simple quantity involving an inverse trigonometric function.)

Exercise 6. Let f denote the majority function for n voters. In class, we showed that $I_i(f) \approx 1/\sqrt{n}$ for all $i \in \{1,\ldots,n\}$. Explain why we can interpret this calculation as saying: your influence in a majority election is a lot more than $1/n$, so you should vote. On the other hand, give reasons why the influence calculation may not accurately reflect your actual influence in a majority election. (If you are thinking of elections in the US, feel free to consider or ignore the electoral college system.)

Exercise 7. Let n be a positive integer. Let $f,g\colon \{-1,1\}^n \to \{-1,1\}$. Let $a_0,\ldots,a_n,b_0,\ldots,b_n \in \mathbf{R}$. Let $x = (x_1,\ldots,x_n) \in \{-1,1\}^n$. For any $x \in \{-1,1\}^n$, define $L_f(x) = a_0 + \sum_{i=1}^n a_i x_i$, $L_g(x) = b_0 + \sum_{i=1}^n b_i x_i$. Assume that $L_f(x) \neq 0$ and $L_g(x) \neq 0$ for all $x \in \{-1,1\}^n$. Assume also that $f(x) = \mathrm{sign}(L_f(x))$ and $g(x) = \mathrm{sign}(L_g(x))$ for all $x \in \{-1,1\}^n$.

Assume that $\widehat{f}(S) = \widehat{g}(S)$ for all $S \subseteq \{1,\ldots,n\}$ with $|S| \leq 1$. Prove that $f = g$. (Hint: what does the Plancherel Theorem say about $\langle f, L_f\rangle$? How does this quantity compare to $\langle g, L_f\rangle$? Also, note that $f(x)L_f(x) = |L_f(x)| \geq g(x)L_f(x)$ for any $x \in \{-1,1\}^n$.)

(Recall $\mathrm{sign}(t) = 1$ if $t > 0$ and $\mathrm{sign}(t) = -1$ if $t < 0$.)

Exercise 8. Let n be a positive integer. Show that there is a one-to-one correspondence (or a bijection) between the set of functions f where $f\colon \{-1,1\}^n \to \mathbf{R}$, and the set of functions g where $g\colon 2^{\{1,2,\ldots,n\}} \to \mathbf{R}$. For example, you could identify a subset $S \subseteq \{1,\ldots,n\}$ with the element $x = (x_1,\ldots,x_n) \in \{-1,1\}^n$ where, for all $i \in \{1,\ldots,n\}$, we have $x_i = 1$ if $i \in S$, and $x_i = -1$ if $i \notin S$.

Let $i,j \in \{1,\ldots,n\}$ and let $x \in \{-1,1\}^n$. Let $S(x) = \{j \in \{1,\ldots,n\} \colon x_j = 1\}$. Using this one-to-one correspondence, show that the i^{th} Shapley value of $f\colon \{-1,1\}^n \to \{-1,1\}$ can be written as

$$\phi_i(f) = \sum_{x \in \{-1,1\}^n \colon x_i = -1} \frac{|S(x)|!(n-|S(x)|-1)!}{n!}\big(f(x_1,\ldots,x_{i-1},1,x_{i+1},\ldots,x_n) - f(x)\big).$$

So, $\phi_i(f)$ is similar to, but distinct from, $I_i(f)$. On the other hand, the i^{th} Banzhaf power index is essentially identical to $I_i(f)$. That is, if we define

$$B_i(f) = \sum_{x \in \{-1,1\}^n} \left| \frac{f(x_1,\ldots,x_{i-1},-x_i,x_{i+1},\ldots,x_n) - f(x)}{2} \right|,$$

Then $B_i(f)/\sum_{j=1}^n B_j(f)$ is the i^{th} Banzhaf power index of f.

Exercise 9. Let $f\colon \{-1,1\}^n \to \{-1,1\}$. Assume that $\widehat{f}(S) = 0$ whenever $S \subseteq \{1,\ldots,n\}$ and $|S| \neq 1$. Show that there exists $i \in \{1,\ldots,n\}$ such that $f(x) = f(x_1,\ldots,x_n) = x_i$ for all $x \in \{-1,1\}^n$, or $f(x) = -x_i$ for all $x \in \{-1,1\}^n$. (This exercise therefore completes the proof of Arrow's Theorem.)