Please provide complete and well-written solutions to the following exercises.

Due April 3, 1159PM PST, to be uploaded as a single PDF document to Brightspace.

Homework 5

Exercise 1. Suppose we have a two-person zero-sum game with $(n+1)\times(n+1)$ payoff matrix A such that at least one entry of A is nonzero. Let $x, y \in \Delta_{n+1}$. Write $x = (x_1, \ldots, x_n, 1 - \sum_{i=1}^{n} x_i)$, $y = (x_{n+1}, x_{n+2}, \ldots, x_{2n}, 1 - \sum_{i=n+1}^{2n} x_i)$. Consider the function $f\colon \mathbf{R}^{2n} \to \mathbf{R}$ defined by $f(x_1, \ldots, x_{2n}) = x^T A y$. Show that the Hessian of f has at least one positive eigenvalue, and at least one negative eigenvalue. Conclude that any critical point of f is a saddle point. That is, if we find a critical point of f (as we sometimes do when we look for the value of the game), then this critical point is a saddle point of f. In this sense, the minimax value occurs at a saddle point of f.

(Hint: Write f in the form $f(x_1, \ldots, x_{2n}) = \sum_{i=1}^{2n} b_i x_i + \sum_{\substack{1 \le i \le n,\\ n+1 \le j \le 2n}} c_{ij} x_i x_j$, where $b_i, c_{ij} \in \mathbf{R}$. From here, it should follow that there exists a nonzero matrix C such that the Hessian of f, i.e. the matrix of second order partial derivatives of f, should be of the form $\begin{pmatrix} 0 & C \\ C^T & 0 \end{pmatrix}$. For simplicity, you are allowed to assume that C is invertible. (This assumption makes the exercise easier, since you should be able to show that the determinant of the Hessian is negative, but the assumption that C is invertible is not actually necessary to complete the exercise.))

Exercise 2. Suppose we have a two-person zero-sum game. Show that any optimal strategy is a Nash equilibrium. Then, show that any Nash equilibrium is an optimal strategy. In summary, the Nash equilibrium generalizes the notion of optimal strategy. (Hint: to prove that a Nash equilibrium is an optimal strategy it may be helpful to argue by contradiction, and to assume that there is a Nash equilibrium that is not an optimal strategy. Then, it may be helpful to use the first part of the argument in our proof of the Minimax Theorem.)

Exercise 3. Show that, in any two-player general-sum game, for any $i \in \{1, 2\}$, the payoffs for player i in any Nash equilibrium exceeds the minimax value for player i. (If A is the $m \times n$ payoff matrix for player 1, then the minimax value for player 1 is the quantity $\max_{x\in\Delta_m} \min_{y\in\Delta_n} x^T A y = \min_{y\in\Delta_n} \max_{x\in\Delta_m} x^T A y$; If B is the $m \times n$ payoff matrix for player 2, then the minimax value for player 2 is the quantity $\max_{y\in\Delta_m} \min_{x\in\Delta_n} x^T B y = \min_{x\in\Delta_n} \max_{y\in\Delta_m} x^T B y$.)

Exercise 4. Recall that the game of Rock-Paper-Scissors is defined by the payoff matrices

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}, \qquad B = A^T.$$

Then the game is symmetric. (And also, note that $A+B=0$, so that the game is a zero-sum game.)

Show that $(1/3, 1/3, 1/3)$ is the unique Nash equilibrium. Then, show that this Nash equilibrium is **not** evolutionarily stable.

This observation leads to interesting behaviors in population dynamics. A certain type of lizard has three kinds of sub-species whose interactions resemble the Rock-Paper-Scissors game. The dynamics of the population cycle between large, dominant sub-populations of each of the three sub-species. That is, first the "Rock" lizards are a majority of the population, then the "Paper" lizards become the majority, then the "Scissors" lizards become the majority, and then the "Rock" lizards become the majority, and so on.

Exercise 5. Let n be a positive integer. Let $v\colon 2^{\{1,\ldots,n\}} \to \{0,1\}$ be a characteristic function that only takes values 0 and 1. Assume also that v is monotonic. That is, if $S,T \subseteq \{1,\ldots,n\}$ with $S \subseteq T$, then $v(S) \leq v(T)$. The **Shapley-Shubik power index** of each player is defined to be their Shapley value.

By monotonicity of v, we have $v(S\cup\{i\}) \geq v(S)$ for all $S \subseteq \{1,\ldots,n\}$ and for all $i \in \{1,\ldots,n\}$. Also, since v only takes values 0 and 1, we have

$$v(S\cup\{i\}) - v(S) = \begin{cases} 1 & \text{when } v(S\cup\{i\}) > v(S) \\ 0 & \text{when } v(S\cup\{i\}) = v(S) \end{cases}.$$

Consequently, we have the following simplified formula for the Shapley-Shubik power index of player $i \in \{1,\ldots,n\}$:

$$\phi_i(v) = \sum_{S\subseteq\{1,\ldots,n\}\colon\, v(S\cup\{i\})=1 \text{ and } v(S)=0} \frac{|S|!(n-|S|-1)!}{n!}.$$

Compute the Shapley-Shubik power indices for all players on the UN security council, with pre-1965 and post-1965 structure. Which structure is better for nonpermanent members?

In pre-1965 rules, the UN security council had five permanent members, and six nonpermanent members. A resolution passes only if all five permanent members want it to pass, and at least two nonpermanent members want it to pass. So, we can model this voting method, by letting $\{1,2,\ldots,11\}$ denote the council, and letting $\{1,2,3,4,5\}$ denote the permanent members. Then we use the characteristic function $v\colon 2^{\{1,\ldots,11\}} \to \{0,1\}$ so that, for any $S \subseteq \{1,\ldots,11\}$, $v(S)=1$ if $\{1,2,3,4,5\} \subseteq S$ and if $|S| \geq 7$. And $v(S)=0$ otherwise.

This voting method was called unfair, so it was restructured in 1965. After the restructuring, the council had the following form (which is still used today). The UN security council has five permanent members, and now ten nonpermanent members. A resolution passes only if all five permanent members want it to pass, and at least four nonpermanent members want it to pass. So, we can model this voting method, by letting $\{1,\ldots,15\}$ denote the council, and letting $\{1,2,3,4,5\}$ denote the permanent members. Then we use the characteristic function $v\colon 2^{\{1,\ldots,15\}} \to \{0,1\}$ so that, for any $S \subseteq \{1,\ldots,15\}$, $v(S)=1$ if $\{1,2,3,4,5\} \subseteq S$ and if $|S| \geq 9$. And $v(S)=0$ otherwise.

Exercise 6. Let n be a positive integer. Let $v\colon 2^{\{1,\ldots,n\}} \to \{0,1\}$ be a characteristic function that only takes values 0 and 1. Assume also that v is monotonic and $v(\{1,\ldots,n\}) = 1$. For each $i \in \{1,\ldots,n\}$, let B_i be the number of subsets $S \subseteq \{1,\ldots,n\}$ such that $v(S) = 0$ and $v(S \cup \{i\}) = 1$. The **Banzhaf power index** of player i is defined to be

$$\frac{B_i}{\sum_{j=1}^{n} B_j}.$$

Like the Shapley-Shubik power index, the Banzhaf power index is another way to measure the relative power of each player.

Compute the Banzhaf power indices for all players for the glove market example.

Then, compute the Banzhaf power indices for all players on the UN security council, with pre-1965 and post-1965 structure. Which structure is better for nonpermanent members?

Exercise 7. Suppose we have two buyers, and $f(v) = 1$ for any $v \in [0,1]$ in a sealed-bid second price auction. That is, V_1 and V_2 are uniformly distributed in the interval $[0,1]$. Show that an equilibrium strategy is $\beta_1(v) = v$, $\beta_2(v) = v$, $\forall\ v \in [0,1]$. That is, each player will bid exactly their private value.

Exercise 8 (**Muddy Children Puzzle/ Blue-Eyed Islanders Puzzle**)**.** This exercise is meant to test our understanding of common knowledge.

Situation 1. There are 100 children playing in the mud. All of the children have muddy foreheads, but any single child cannot tell whether or not her own forehead is muddy. Any child can also see all of the other 99 children. The children do not communicate with each other in any way, there are no mirrors or recording devices, etc. so that no child can see her own forehead. The teacher now says, "stand up if you know your forehead is muddy." No one stands up, because no one can see her own forehead. The teacher asks again. "Knowing that no one stood up the last time, stand up now if you know your forehead is muddy." Still no one stands up. No matter how many times the teacher repeats this statement, no child stands up.

Situation 2. After Situation 1, the teacher now says, "I announce that at least one of you has a muddy forehead." The teacher then says, "stand up if you know your forehead is muddy." No one stands up. The teacher pauses then repeats, "stand up if you know your forehead is muddy." Again, no on stands up. The teacher continues making this statement. The hundredth time that she makes this statement, all the children suddenly stand up.

Explain why all of the children stand up in Situation 2, but they do not stand up in Situation 1. Pay close attention to what is common knowledge in each situation.

Exercise 9. There are five pirates on a ship. It is also common knowledge that every pirate prefers to maximize their amount of gold. There are 100 gold pieces to be split amongst the pirates. The game begins when the first pirate proposes how they think the gold should be split amongst the five pirates. All five pirates vote whether or not to accept the proposal, by a majority vote. If the proposal is accepted, the game ends. If the proposal is not accepted, the first pirate is thrown overboard, and the game begins continues. The second pirate now proposes how they think the gold should be split amongst the four remaining pirates. All four pirates vote whether or not to accept the proposal, by a majority vote (the current proposer, i.e. the second pirate breaks a tie). If the proposal is accepted, the game ends. If the proposal is not accepted, the second pirate is thrown overboard, and the game continues, etc. (During any voting phase, if a pirate's share of gold will decrease by throwing the proposer overboard, this pirate will vote to accept the proposal; otherwise this pirate will vote to not accept the proposal.) What is the largest amount of gold that the first pirate can obtain in the game?

Exercise 10. Explain what a buyer in an open-bid decreasing auction knows when the current announced price is x that they did not know prior to the start of the auction. (What is common knowledge?)