Please provide complete and well-written solutions to the following exercises.

Due March 13, 1159PM PST, to be uploaded as a single PDF document to Brightspace.

Homework 4

Exercise 1. Show the following fact, which will be mentioned after our proof of Sperner's Lemma:

Let d be a positive integer. Let K be a closed and bounded subset of $\mathbf{R}^d$. Then the set $K \times K$ is also a closed and bounded set.

(Recall that $K \times K = \{(x, y) \in \mathbf{R}^d \times \mathbf{R}^d : x \in K \text{ and } y \in K\} \subseteq \mathbf{R}^{2d}$.)

Exercise 2. Show the following facts, which will be used in our discussion of Correlated equilibria:

For any $x, y \in \Delta_2$, $xy^T \neq \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$.

For any $x, y \in \Delta_2$, $xy^T \neq \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & 0 \end{pmatrix}$.

Exercise 3. Recall the prisoner's dilemma, which is described by the following payoffs

Prisoner I	Prisoner II: silent	Prisoner II: confess
silent	$(-1, -1)$	$(-10, 0)$
confess	$(0, -10)$	$(-8, -8)$

Recall that this two-person game has exactly one Nash equilibrium, where both parties confess. However, if this game is repeated an infinite number of times, or a random number of times, this strategy is no longer the only Nash equilibrium. This exercise explores the case where the game is repeated an infinite number of times. Let N be a positive integer. Suppose the game is repeated infinitely many times, so that player I has payoffs $a_1, a_2, a_3, \ldots$ and player II has payoffs $b_1, b_2, b_3, \ldots$. That is, at the i^{th} iteration of the game, player I has payoff a_i and player II has payoff b_i. In the infinitely repeated game, each player would like to maximize her average payoff over time (if this average exists). That is, player I wants to maximize $\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} a_i$ and player II wants to maximize $\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} b_i$.

Consider the following strategy for player I. Suppose player I begins by staying silent, and she continues to be silent on subsequent rounds of the game. However, if player II confesses at round $i \geq 1$ of the game, then player I will always confess for every round of the game after round i. Player II follows a similar strategy. Suppose player II begins by staying silent, and she continues to be silent on subsequent rounds of the game. However, if player I confesses at round $j \geq 1$ of the game, then player II will always confess for every round of the game after round j.

Show that this pair of strategies is a Nash equilibrium. That is, no player can gain something by unilaterally deviating from this strategy.

Exercise 4. Show that the following strategy (known as "quid pro quo") is also a Nash equilibrium for Prisoner's Dilemma iterated an infinite number of times.

Player I begins by staying silent. If Player II plays x on round i, then Player I plays x on round $i+1$. Similarly, Player II begins by staying silent. If Player I plays x on round i, then Player II plays x on round $i+1$.

Exercise 5. Find all Correlated Equilibria for the Prisoner's Dilemma.

Exercise 6. We return now to the setting of general sum games. Show that any convex combination of Nash equilibria is a Correlated Equilibrium. That is, if $z(1),\ldots,z(k)$ are Nash Equilibria, and if $t_1,\ldots,t_k \in [0,1]$ satisfy $\sum_{i=1}^{k} t_i = 1$, then $\sum_{i=1}^{k} t_i z(i)$ is a Correlated Equilibrium.

Exercise 7. Recall the Game of Chicken is defined as follows. Each player chooses to chicken out (C) by swerving away, or she can continue drive straight (D). Each player would prefer to continue driving while the other chickens out. However, if both players choose to continue driving, catastrophe occurs. The payoffs follow:

Player I \ Player II	C	D
C	$(6,6)$	$(2,7)$
D	$(7,2)$	$(0,0)$

Find all Nash equilibria for the Game of Chicken. Prove that these are the only Nash equilibria. Then, verify that

$$z = \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & 0 \end{pmatrix}$$

is a Correlated Equilibrium. Can you find a Correlated Equilibrium such that both players have a payoff larger than 5? (Hint: when trying to find such a matrix z, assume that $z_{22} = 0$ and $z_{12} = z_{21}$.)

Exercise 8. In the Game of Chicken, you should have found only three Nash equilibria. Recall that any convex combination of Nash equilibria is a correlated equilibrium. However, the converse is false in general! We can see this already in the Game of Chicken. Show that the Correlated Equilibrium

$$z = \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & 0 \end{pmatrix}$$

is not a convex combination of the Nash equilibria. Put another way, the payoffs from this Correlated Equilibrium cannot be found by randomly choosing among the Nash equilibria.

Exercise 9. Give an example of a two-person zero-sum game where there are no pure Nash equilibria. Can you give an example where all entries of the payoff matrix are different?