Please provide complete and well-written solutions to the following exercises.

Due February 27, 1159PM PST, to be uploaded as a single PDF document to Brightspace.

Homework 3

Exercise 1. This exercise deals with subsets of the real line. Show that $[0,1]$ is closed, but $(0,1)$ is not closed.

Exercise 2. This exercise deals with subsets of Euclidean space $\mathbf{R}^d$ where $d \geq 1$. Show that the intersection of two closed sets is a closed set.

Exercise 3. Define $f\colon \mathbf{R}^d \to \mathbf{R}$ by $f(x) := ||x||$. Show that f is continuous. (Hint: you may need to use the triangle inequality, which says that $||x+y|| \leq ||x|| + ||y||$, for any $x, y \in \mathbf{R}^d$. Also, recall that $||(x_1, \ldots, x_d)|| = (\sum_{i=1}^d x_i^2)^{1/2}$.)

Exercise 4. Describe in words the set of points (x_1, x_2) in the plane such that $(x_1, x_2) \geq (3, 4)$.

Exercise 5. Let d be a positive integer. Consider

$$\Delta_d := \{x = (x_1, \ldots, x_d) \in \mathbf{R}^d \colon \sum_{i=1}^d x_i = 1,\, x_i \geq 0,\ \forall\, 1 \leq i \leq d\}.$$

Prove that Δ_d is convex, closed and bounded.

Exercise 6.

- Let K be the set of points (x, y) in the plane such that $|x| + |y| \leq 2$. Is K convex? Prove your assertion.
- Let K be the set of points (x, y, z) in $\mathbf{R}^3$ such that $\max(|x|, |y|, |z|) \leq 1/2$. Is K convex? Prove your assertion.
- Let K be the set of points (x, y, z, w) in $\mathbf{R}^4$ such that $x^2 + y^2 + z^2 + w^2 \leq 1$. You may assume that K is convex. Find a hyperplane that separates K from the point $(0, 1, 1, 0)$.

Exercise 7. Show that the intersection of two convex sets is convex. Then, show that the intersection of any finite number of convex sets is convex. Finally, find two convex sets A, B such that the union $A \cup B$ is not convex.

Exercise 8. Let A be an $n \times m$ real matrix. Let $b \in \mathbf{R}^n$, $c \in \mathbf{R}^m$. Using the Minimax Theorem, prove the following equality, which is known as duality for linear programming:

$$\min_{x \in \mathbf{R}^m \colon Ax \geq b,\, x \geq 0} x^T c = \max_{y \in \mathbf{R}^n \colon A^T y \leq c,\, y \geq 0} b^T y$$

(Hint: Consider the game with $(n+m+1)\times(n+m+1)$ payoff matrix given by

$$\begin{pmatrix} 0 & A & -b \\ -A^T & 0 & c \\ b^T & -c^T & 0 \end{pmatrix}.$$

First, show that the value of the game is 0. Then, apply the Minimax Theorem to this payoff matrix. Using Exercise 9, conclude there exists $x \in \mathbf{R}^m, y \in \mathbf{R}^n, t \in \mathbf{R}$ such that $\sum_{i=1}^m x_i + \sum_{i=1}^n y_i + t = 1$, $x \geq 0$, $y \geq 0$, $t \geq 0$, and such that

$$\begin{pmatrix} 0 & A & -b \\ -A^T & 0 & c \\ b^T & -c^T & 0 \end{pmatrix} \begin{pmatrix} y \\ x \\ t \end{pmatrix} \geq 0.$$

In particular, $b^T y - c^T x \geq 0$. As a simplifying assumption, you may assume $t > 0$. Then, x/t and y/t achieve the minimum and maximum values, respectively, in the duality for linear programming. To show this, prove the following claim. For any $x \in \mathbf{R}^m$ with $Ax \geq b$ and for any $y \in \mathbf{R}^n$ with $A^T y \leq c$, where $x \geq 0, y \geq 0$, we have $c^T x - b^T y \geq 0$.)

Consider now an example where $n = m = 2$, $b = (1,0)$, $c = (1,1)$ and $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Using the duality above, show that

$$\max_{y \in \mathbf{R}^n :\, A^T y \leq c,\, y \geq 0} b^T y \leq 1.$$

Exercise 9. Let $x \in \Delta_m$, $y \in \Delta_n$ and let A be an $m \times n$ matrix. Show that

$$\max_{x \in \Delta_m} x^T A y = \max_{i=1,\ldots,m} (Ay)_i, \qquad \min_{y \in \Delta_n} x^T A y = \min_{j=1,\ldots,n} (x^T A)_j.$$

Using this fact, show that

$$\min_{y \in \Delta_n} \max_{x \in \Delta_m} x^T A y = \min_{y \in \Delta_n} \max_{i=1,\ldots,m} (Ay)_i.$$

$$\max_{x \in \Delta_m} \min_{y \in \Delta_n} x^T A y = \max_{x \in \Delta_m} \min_{j=1,\ldots,n} (x^T A)_j.$$

Using the second equality, conclude that the value of the game with payoff matrix A can be found via the following Linear Programming problem:

Maximize t subject to the constraints: $\sum_{i=1}^m x_i a_{ij} \geq t$, for all $1 \leq j \leq n$; $\sum_{i=1}^m x_i = 1$; $x \geq (0,\ldots,0)$.

Efficient methods for solving linear programming problems are well-known. However, below we will focus on ways to compute the values of two-person zero-sum games by hand.

(Hint: it might be better to write the first constraint as $x^T A \geq (t,\ldots,t)$.)

Exercise 10. Find the value of the two-person zero-sum game described by the payoff matrix

$$\begin{pmatrix} 0 & 9 & 1 & 1 \\ 5 & 0 & 6 & 7 \\ 2 & 4 & 3 & 3 \end{pmatrix}$$

Exercise 11. Find the value of the two-person zero-sum game described by the payoff matrix

$$\begin{pmatrix} 0 & 7 & 0 & 6 \\ 4 & 4 & 3 & 3 \\ 8 & 2 & 6 & 0 \end{pmatrix}$$

Exercise 12. This Exercise shows that von Neumann's Minimax Theorem no longer holds when we consider games for three or more players.

first, note that there is a suitable generalization of this theorem to two-player general-sum games. That is if A is the payoff matrix for player I and B is the payoff matrix for player II, then

$$\max_{x\in\Delta_m}\min_{y\in\Delta_n} x^T Ay = \min_{y\in\Delta_n}\max_{x\in\Delta_m} x^T Ay.$$
$$\max_{x\in\Delta_m}\min_{y\in\Delta_n} x^T By = \min_{y\in\Delta_n}\max_{x\in\Delta_m} x^T By.$$

In words, the first equality says: the maximum over player I's strategies followed by the minimum of the other players strategies of the payoff of player I is equal to the minimum of the other players strategies followed by the maximum over player I's strategies of the payoff of player I.

Now, consider a three-player general-sum game. The analogue of von Neumann's Theorem just applied to player I would say: the maximum over player I's strategies followed by the minimum of the other players strategies of the payoff of player I is equal to the minimum of the other players strategies followed by the maximum over player I's strategies of the payoff of player I.

Show that this statement is false for the following example.

	L	R
T	0	1
B	1	1

W

	L	R
T	1	1
B	1	0

E

These matrices describe the payoffs for player I. In the game, player I chooses a row (T or B), player II chooses a column (L or R), and player III chooses a matrix (W or E)