Please provide complete and well-written solutions to the following exercises.

Due February 6, 1159PM PST, to be uploaded as a single PDF document to Brightspace.

Homework 2

Exercise 1. Compute the following nim-sums: $3 \oplus 4$, $5 \oplus 9$. Then, let a, b, c be nonnegative integers. Prove that $a \oplus a = 0$ and $(a \oplus b) \oplus 0 = a \oplus b$.

Exercise 2. Consider the nim position $(9, 10, 11, 12)$. Which player has a winning strategy from this position, the next player or the previous player? Describe a winning first move.

Exercise 3. Let G_1, G_2 be games. Let x_i be a game position for G_i, and let $\mathbf{N}_{G_i}, \mathbf{P}_{G_i}$ denote, $\mathbf{N}$ and $\mathbf{P}$ respectively for the game G_i, for each $i \in \{1, 2\}$. Show the following:

(i) If $x_1 \in \mathbf{P}_{G_1}$ and if $x_2 \in \mathbf{P}_{G_2}$, then $(x_1, x_2) \in \mathbf{P}_{G_1+G_2}$.
(ii) If $x_1 \in \mathbf{P}_{G_1}$ and if $x_2 \in \mathbf{N}_{G_2}$, then $(x_1, x_2) \in \mathbf{N}_{G_1+G_2}$.
(iii) If $x_1 \in \mathbf{N}_{G_1}$ and if $x_2 \in \mathbf{N}_{G_2}$, then (x_1, x_2) could be in either $\mathbf{N}_{G_1+G_2}$ or $\mathbf{P}_{G_1+G_2}$.

Exercise 4. Let G_1, G_2, G_3 be games. Show that the notion of two games being equivalent is an equivalence relation. That is, show the following

- G_1 is equivalent to G_1.
- If G_1 is equivalent to G_2, then G_2 is equivalent to G_1.
- If G_1 is equivalent to G_2, and if G_2 is equivalent to G_3, then G_1 is equivalent to G_3.

Exercise 5. Show that in the game of chess, exactly one of the following situations is true:

- White has a winning strategy.
- Black has a winning strategy.
- Each of the two players has a strategy guaranteeing at least a draw.

You may assume that chess is progressively bounded. (Hint: you should not really need to use anything special about chess, other than that it is a partisan combinatorial game that is progressively bounded. Also, as usual, it is probably beneficial to start from a terminal position, and then work backwards, using induction.)

Exercise 6. We first describe the game of Y. In this game, there is an arrangement of white hexagons in an equilateral triangle. One player is assigned the color blue, and the other player is assigned the color yellow. The players then take turns filling in one hexagon at a time of their assigned color. The goal is to create a Y-shape that connects all three sides of the triangle. That is, the goal of the game is to have an unbroken path of a single color of hexagons that touches all three sides of the triangle.

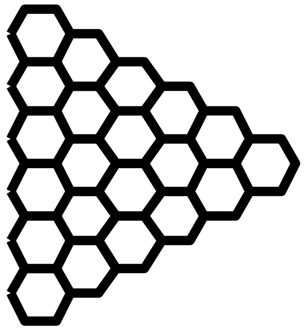

Figure 1. A Starting Position in the game of Y

Prove that the game of Hex can be realized as a special case of the game of Y. That is, the opening position on a standard hex board is equivalent to a particular game position in the game of Y. (Recall that we defined a notion for two games being equivalent.)

Exercise 7. Describe the optimal strategies for both players in rock-paper-scissors. Prove that these strategies are optimal. This game is described by the following payoff matrix.

		Player II		
		R	P	S
Player I	R	0	-1	1
	P	1	0	-1
	S	-1	1	0

Exercise 8. Describe the optimal strategies for both players for the two-person zero-sum game described by the payoff matrix

		Player II	
		A	B
Player I	C	0	2
	D	4	1

Prove that these strategies are optimal.