454 Final Solutions[1]

1. Question 1

TRUE/FALSE

(a) In the game of Nim, the first player always has a winning strategy.

FALSE. If the game position is $(1,1)$, i.e. two piles of equal size, then player 1 will lose since player 2 must take the last chip.

(b) Consider the problem of deciding if a general sum game has at least two Nash equilibria. This problem is NP-complete.

True. We mentioned this in class.

(c) There exists a symmetric two-person general-sum game such that all of its Nash equilibria are not symmetric. (The $n\times n$ payoff matrices A,B satisfy $A=B^T$, and all equilibria $(x,y)\in\Delta_n\times\Delta_n$ satisfy $x\neq y$.)

FALSE. Every symmetric game has at least one symmetric Nash equilibrium.

(d) Let $f\colon\{-1,1\}^n\to\{-1,1\}$ be a function. Suppose we do a Condorcet election with f (so that if we just look at the votes between any pair of two candidates, the aggregate preference is decided using the function f). Suppose a Condorcet winner always exists. Then there exists $i\in\{1,\ldots,n\}$ such that $f(x_1,\ldots,x_n)=x_i$ for all $(x_1,\ldots,x_n)\in\{-1,1\}^n$.

FALSE. It could occur that $f(x_1,\ldots,x_n)=-x_i$ for all $(x_1,\ldots,x_n)\in\{-1,1\}^n$.

(e) Let $f,g\colon\{-1,1\}^n\to\{-1,1\}$ be functions. Assume that n is odd and f is the majority function ($f(x_1,\ldots,x_n)=\mathrm{sign}(x_1+\cdots+x_n)$). Assume $f\neq g$. Then for any $\rho\in(0,1)$, the noise stability of f exceeds the noise stability of g.

FALSE. the constant function $f=1$ has the largest noise stability among all such functions.

2. Question 2

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Recall the prisoner's dilemma, which has the following payoffs.

Prisoner I \ Prisoner II	silent	confess
silent	$(-1,-1)$	$(-10,0)$
confess	$(0,-10)$	$(-8,-8)$

Find all Nash equilibria for this game.

Solution. As shown in the notes, it follows from a domination argument that $x=(0,1)$ and $y=(0,1)$ is the only Nash equilibrium for this game.

3. Question 3

(a) Give an example of a closed and convex subset K of Euclidean space, and give an example of a continuous function $f\colon K\to K$ such that f has no fixed point.

(b) Give an example of a bounded and closed subset K of Euclidean space, and give an example of a continuous function $f\colon K\to K$ such that f has no fixed point.

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(c) Give an example of a function $f\colon \mathbf{R}\to\mathbf{R}$ such that the only fixed point of f is the point $x=1$.

Solution. For (a) we can use $f\colon \mathbf{R}\to\mathbf{R}$ with $f(x)=x+1$ for all $x\in\mathbf{R}$. A fixed point means $x=f(x)=x+1$, i.e. $0=1$, a contradiction. Evidently $\mathbf{R}$ is closed and convex and f is continuous since it is linear.

For (b), Let $K\subseteq\mathbf{R}^2$ be the unit circle and let $f\colon K\to\ K$ be a rotation of K by an angle $\pi/2$.

For (c), let $f(x)=1$ for all $x\in\mathbf{R}$. Then $x=f(x)$ implies $x=1$, i.e. $x=1$ is the only fixed point of f.

4. Question 4

Ten people are standing together in a room. They are presented with the following problem: each person chooses a real number between (and including) 0 and 100. (So, someone could guess: 20, 51.5, π, $\sqrt{2}$, 99.999, etc.) The person who chooses the number closest to two-thirds of the average of all of the numbers wins. That is, if

$$0\leq a_1,\ldots,a_{10}\leq 100,$$

each person wants to choose a number closest to $(2/3)(1/10)\sum_{i=1}^{10}a_i$. The people do not communicate with each other in any way. It is common knowledge that every person wants to win the game, and every person is rational. Explain what number each person will choose.

Solution. Every person will choose 0.

To see why, consider the actions of the first person. Since every bid will be at most 100, we have

$$(2/3)(1/10)\sum_{i=1}^{10}a_i\leq(2/3)(1/10)\sum_{i=1}^{10}100=(2/3)100.$$

That is, the largest bid any person will rationally choose is (2/3)100. However, all players are aware of this fact, so all players know everyone will rationally bid at most (2/3)100. Consequently,

$$(2/3)(1/10)\sum_{i=1}^{10}a_i\leq(2/3)(1/10)\sum_{i=1}^{10}(2/3)100=(2/3)^2 100.$$

So, the largest bid any person will rationally choose is $(2/3)^2100$. And all players are aware of this fact. And so on.

In general, if there is some upper bound T with $0\leq T\leq 100$ on all guesses that all players will make, and if T is common knowledge, then the largest bid any rational person will consider making is

$$(2/3)(1/10)\sum_{i=1}^{10}a_i\leq(2/3)(1/10)\sum_{i=1}^{10}T=(2/3)T.$$

And this fact is also common knowledge. So, the largest bid any person will rationally make is $\lim_{N\to\infty}(2/3)^N T=0$.

5. Question 5

Explain in detail the Condorcet voting paradox. You should probably use an example of three voters ranking three candidates in order to explain the paradox.

Solution. Consider the following ranking of three candidates a, b, c between three voters $1, 2, 3$.

Voter	Rank 1	Rank 2	Rank 3
1	a	b	c
2	b	c	a
3	c	a	b

If we ignore candidate b, then voters 2 and 3 prefer c over a, while voter 1 prefers a over c. So, using a majority rule for these preferences, the voters prefer c over a. If we ignore candidate c, then voters 1 and 3 prefer a over b, while voter 2 prefers b over a. So, using a majority rule again, the voters prefer a over b. Finally, if we ignore candidate a, then voters 1 and 2 prefer b over c, while voter 3 prefers c over b. So, using a majority rule, the voters prefer b over c.

In conclusion, given the above rankings, if we use a majority rule for every comparison between two candidates, the voters prefer a over b, they prefer b over c, and they prefer c over a. So, no one has won the election, if we conduct a Condorcet election.

6. Question 6

Let n be a positive integer. Let $v\colon 2^{\{1,\ldots,n\}} \to \mathbf{R}$ be a characteristic function (so that $v(\emptyset) = 0$). For any $1 \leq i \leq n$, define

$$B_i(v) := 2^{-n} \sum_{S \subseteq \{1,\ldots,n\}\colon\, i \in S} |v(S) - v(S \smallsetminus \{i\})|.$$

(Here $S \smallsetminus \{i\} = \{s \in S\colon s \neq i\}$.)

- Which of Shapley's four axioms do the values $(B_1, \ldots, B_n)$ satisfy?
  JUSTIFY YOUR ANSWER.
- Is B_i equal to the i^{th} Shapley value for each $1 \leq i \leq n$?
  JUSTIFY YOUR ANSWER.

Solution. We can check through examples that the symmetry and no-power/no-value axioms hold, but the other two do not hold. Since not all four axioms hold, the B_i are not equal to the Shapley values, by Shapley's theorem. (There is a unique solution satisfying the four axioms.)

More specifically: the first symmetry axiom holds since, for any $1 \leq i < j \leq n$, we have

$$
\begin{aligned}
B_i(v) &:= 2^{-n} \sum_{S \subseteq \{1,\ldots,n\}\colon i \in S} |v(S) - v(S \setminus \{i\})| \\
&= 2^{-n} \sum_{S \subseteq \{1,\ldots,n\}\colon i,j \in S} |v(S) - v(S \setminus \{i\})| + 2^{-n} \sum_{S \subseteq \{1,\ldots,n\}\colon i \in S, j \notin S} |v(S) - v(S \setminus \{i\})| \\
&= 2^{-n} \sum_{S \subseteq \{1,\ldots,n\}\colon i,j \in S} |v(S) - v(S \setminus \{i\})| + 2^{-n} \sum_{S \subseteq \{1,\ldots,n\}\colon i \in S, j \notin S} |v(S \setminus \{i\} \cup \{i\}) - v(S \setminus \{i\})| \\
&\overset{(*)}{=} 2^{-n} \sum_{S \subseteq \{1,\ldots,n\}\colon i,j \in S} |v(S) - v(S \setminus \{j\})| + 2^{-n} \sum_{S \subseteq \{1,\ldots,n\}\colon i \in S, j \notin S} |v(S \setminus \{i\} \cup \{j\}) - v(S \setminus \{i\})| \\
&= 2^{-n} \sum_{S \subseteq \{1,\ldots,n\}\colon i,j \in S} |v(S) - v(S \setminus \{j\})| + 2^{-n} \sum_{S \subseteq \{1,\ldots,n\}\colon j \in S, i \notin S} |v(S) - v(S \setminus \{j\})| \\
&= 2^{-n} \sum_{S \subseteq \{1,\ldots,n\}\colon j \in S} |v(S) - v(S \setminus \{j\})| = B_j(v).
\end{aligned}
$$

Here $(*)$ used the symmetry assumption that $v(S \cup \{i\}) = v(S \cup \{j\})$ for any $S \subseteq \{1,\ldots,n\}$ with $i, j \notin S$.

The second no power/no value axiom holds essentially by definition, since if $v(S \cup \{i\}) = v(S)$ for any $S \subseteq \{1,\ldots,n\}$, then $v(S) = v(S \setminus \{i\})$ for any $S \subseteq \{1,\ldots,n\}$, i.e. $B_i(v) = 0$ since it is a sum of zeros.

The third additivity assumption does not hold. Consider $v\colon 2^{\{1,2\}} \to \mathbf{R}$ defined by $v(S) = 0$ for all $S \subseteq \{1,2\}$ except $v(\{1\}) = 1$, and let $u\colon 2^{\{1,2\}} \to \mathbf{R}$ defined by $u(S) = 0$ for all $S \subseteq \{1,2\}$ except $u(\{1,2\}) = 1$. Then

$$
\begin{aligned}
B_1(v) &= 2^{-2}(|v(\{1,2\}) - v(1)| + |v(1) - v(\emptyset)|) = 2^{-2}(2) = 1/2 \\
B_2(v) &= 2^{-2}(|v(\{1,2\}) - v(2)| + |v(2) - v(\emptyset)|) = 0 \\
B_1(u) &= 2^{-2}(|u(\{1,2\}) - u(1)| + |u(1) - u(\emptyset)|) = 2^{-2}(1) = 1/4 \\
B_2(u) &= 2^{-2}(|u(\{1,2\}) - u(2)| + |u(2) - u(\emptyset)|) = 2^{-2}(1) = 1/4
\end{aligned}
$$

Note that $(u + v)(S) = 1$ when $S = \{1\}$ or $S = \{1,2\}$, but otherwise $u + v = 0$. So,

$$
\begin{aligned}
B_1(u + v) &= 2^{-2}(1) = 1/4 \\
B_2(u + v) &= 2^{-2}(1) = 1/4.
\end{aligned}
$$

So, $B_1(u + v) \neq B_1(u) + B_1(v)$.

The fourth efficiency axiom does not hold. From the previous example, we have $u(\{1,2\}) = 1$ while $B_1(u) = B_2(u) = 1/4$, so $u(\{1,2\}) \neq B_1(u) + B_2(u)$.

Finally, since only two of the four axioms hold, $(B_1, \ldots, B_n)$ cannot be equal to the Shapley values, since Shapley's Theorem says that the Shapley values are the unique solution satisfying all four of Shapley's axioms.

7. Question 7

- Describe in detail the Upper Confidence Bound Algorithm.

The input for this algorithm is a bandit problem with k arms. The output for this algorithm is a policy (strategy) for the bandit problem.

- Describe in detail a bound for the expected regret r_H of this algorithm, when the rewards satisfy $0 \leq R_t \leq 1$ for a k-arm bandit problem with horizon H. (You do NOT have to prove this regret bound holds, just state what the bound is.)

Hint: let $N_t(a)$ be the number of times action $a \in \mathcal{A}$ has been played at time t, let $M_{t,a}$ be the sample mean of the rewards obtained at time t from action a being played. For any $t \geq 0$ where $N_t(a) \geq 1$, define

$$UCB_t(a) := M_{t,a} + \sqrt{\frac{2\log H}{N_t(a)}}.$$

Solution. This description is provided in the notes. After trying each action, choose the action that maximizes $UCB_t(a)$ for each subsequent round. The expected regret is then bounded by $10(kH\log H)^{1/2}$, as shown in the notes.

8. Question 8

Suppose we have a two-player symmetric game such with payoffs described by a matrix such as the following.

$$A = \begin{pmatrix} 4 & 3 & 2 & 5 & 6 \\ 3 & 1 & 8 & 9 & 1 \\ 7 & 0 & 7 & 0 & 7 \\ 1 & 3 & 3 & 2 & 1 \\ 0 & 1 & 2 & 9 & 1 \end{pmatrix}$$

Describe in detail an algorithm that outputs a Nash equilibrium of this game.

Make sure to justify why your algorithm outputs a Nash equilibrium.

Also give a bound on the run time of the algorithm.

Solution. We are given the claim: if $(\widetilde{x}, \widetilde{y}) \in \Delta_m \times \Delta_n$ is a Nash equilibrium and if

$$I := \{1 \leq i \leq m : \widetilde{x}_i > 0\}, \qquad J := \{1 \leq j \leq n : \widetilde{y}_j > 0\}. \qquad (*)$$

then with payoff matrices A, B respectively, we have

$$\max_{i=1,\ldots,m} (A\widetilde{y})_i = (A\widetilde{y})_i, \quad \forall i \in I. \qquad \text{and} \qquad \max_{j=1,\ldots,n} (\widetilde{x}^T B)_j = (\widetilde{x}^T B)_j, \forall j \in J.$$

It then follows from the claim that

$$\widetilde{x}^T A\widetilde{y} = (A\widetilde{y})_i, \quad \forall\, i \in I. \qquad \widetilde{x}^T B\widetilde{y} = (\widetilde{x}^T B)_j, \quad \forall\, j \in J.$$

Algorithm 1 Input: $m \times n$ matrices A, B, $I \subseteq \{1, \ldots, m\}$, $J \subseteq \{1, \ldots, n\}$ Assumption: there exists a Nash equilibrium $\widetilde{x} \in \Delta_m, \widetilde{y} \in \Delta_n$ with I, J satisfying $(*)$. Output: $\widetilde{x}, \widetilde{y}$.

- Let $\widetilde{x}_i, i \in I$ and let $\widetilde{y}_j, j \in J$ be the $|I| + |J|$ unknown variables.
- Consider $|I| - 1$ equations $(A\widetilde{y})_i = (A\widetilde{y})_{i'} \quad \forall i, i' \in I$.
- Consider $|J| - 1$ equations $(\widetilde{x}^T B)_j = (\widetilde{x}^T B)_{j'} \quad \forall j, j' \in J$.
- Together with the equations $\sum_{i\in I} \widetilde{x}_i = 1$, $\sum_{j\in J} \widetilde{y}_j = 1$, we have $|I| + |J|$ linear equations in $|I| + |J|$ equations. By assumption, there exists at least one positive solution to these equations. Output such a solution, if one can be found where $\widetilde{x}, \widetilde{y}$ have nonnegative values and they also maximize the expected payoffs of the players (i.e. if $\widetilde{x}^T A\widetilde{y} \geq (A\widetilde{y})_i$ for all $1 \leq i \leq m$ and $\widetilde{x}^T A\widetilde{y} \geq (\widetilde{x}^T B)_j$ for all $1 \leq j \leq n$.)

Algorithm 2 (Finding a Nash Equilibrium). Input: $m \times n$ matrix A, B. Output: $\widetilde{x}, \widetilde{y}$ a Nash equilibrium

- Run Algorithm 1 for every possible pair of subsets $I \subseteq \{1, \ldots, m\}, J \subseteq \{1, \ldots, n\}$.
- From Nash's Theorem, there exists at least one pair I, J where Algorithm 1 succeeds.

From the above discussion, Algorithm 1 finds a Nash equilibrium, if it exists with a given support as in $(*)$. So, Algorithm 2 finds at least one Nash equilibrium, since it searches over all possible supports I, J defined in $(*)$. Moreover, Algorithm 2 has run time at least 2^{m+n}, since we might need to check this many pairs of I, J. If each solution of the linear equations takes, say $10(m+n)^3$ time, (say using Gaussian elimination on at most $m+n$ equations.) then the total run time is at most $10(m+n)^3 2^{m+n}$.