Math 499, Spring 2025, USC — Instructor: Steven Heilman

Name: ________________ USC ID: ____________ Date: __________

Signature: ________________. Discussion Section: ________________
(By signing here, I certify that I have taken this test while refraining from cheating.)

Final Exam

This exam contains 13 pages (including this cover page) and 8 problems. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- You have 50 minutes to complete the exam, starting at the beginning of class.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Do not write in the table to the right. Good luck![a]

Problem	Points	Score
1	18	
2	10	
3	10	
4	10	
5	10	
6	10	
7	15	
8	10	
Total:	93	

Reference sheet. Below are some definitions that may be relevant.

$$\Delta_m := \{x = (x_1, \ldots, x_m) \in \mathbf{R}^m \colon \sum_{i=1}^{m} x_i = 1,\ x_i \geq 0,\ \forall\, 1 \leq i \leq m\}.$$

Let A be an $m \times n$ real payoff matrix defining a zero-sum two-player game. A mixed strategy $\widetilde{x} \in \Delta_m$ is **optimal for player** I if $\min_{y \in \Delta_n} \widetilde{x}^T A y = \max_{x \in \Delta_m} \min_{y \in \Delta_n} x^T A y$. A mixed strategy $\widetilde{y} \in \Delta_n$ is **optimal for player** II if $\max_{x \in \Delta_m} x^T A \widetilde{y} = \min_{y \in \Delta_n} \max_{x \in \Delta_m} x^T A y$. We say the pair $(\widetilde{x}, \widetilde{y})$ are an **optimal strategy** for the payoff matrix A if $\widetilde{x} \in \Delta_m$ is optimal for player I and $\widetilde{y} \in \Delta_n$ is optimal for player II.

Let m, n be positive integers. Suppose we have a two-player general sum game with $m \times n$ payoff matrices. Let A be the payoff matrix for player I and let B be the payoff matrix for player II. A pair of vectors $(\widetilde{x}, \widetilde{y})$ with $\widetilde{x} \in \Delta_m$ and $\widetilde{y} \in \Delta_n$ is a **Nash equilibrium** if

$$\widetilde{x}^T A \widetilde{y} \geq x A \widetilde{y}, \quad \forall\, x \in \Delta_m, \qquad \widetilde{x}^T B \widetilde{y} \geq \widetilde{x} B y, \quad \forall\, y \in \Delta_n.$$

A joint distribution of strategies is an $m \times n$ matrix $z = (z_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ such that $z_{ij} \geq 0$ for all $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$, and such that $\sum_{i=1}^{m} \sum_{j=1}^{n} z_{ij} = 1$. We say z is a **correlated equilibrium** if

$$\sum_{j=1}^{n} z_{ij} a_{ij} \geq \sum_{j=1}^{n} z_{ij} a_{kj}, \qquad \forall\, i \in \{1, \ldots, m\}, \forall\, k \in \{1, \ldots, m\}.$$

$$\sum_{i=1}^{m} z_{ij} b_{ij} \geq \sum_{i=1}^{m} z_{ij} b_{ik}, \qquad \forall\, j \in \{1, \ldots, n\}, \forall\, k \in \{1, \ldots, n\}.$$

Suppose we have a two-player symmetric game (so that the payoff matrix for player I is A, the payoff matrix for player II is B, and with $A = B^T$). Assume that A, B are $n \times n$ matrices. A mixed strategy $x \in \Delta_n$ is said to be an **evolutionarily stable strategy (ESS)** if, for any pure strategy w, we have

$$w^T A x \leq x^T A x, \qquad \text{and} \qquad \text{If } w^T A x = x^T A x, \text{ then } w^T A w < x^T A w.$$

A set $K \subseteq \mathbf{R}^n$ is called **convex** if, for any $x, y \in K$ and for any $0 \leq t \leq 1$, we have $tx + (1-t)y \in K$. A set $K \subseteq \mathbf{R}^n$ is called **bounded** if there exists $r > 0$ such that $||x|| \leq r$ for all $x \in K$.

We define a **symmetric auction**. A single object is for sale at an auction. The seller is willing to sell the object at any nonnegative price. There are n buyers, which we identify with the set $\{1, 2, \ldots, n\}$. All buyers have some set of **private values** in $[0, \infty)$. We denote the private value of buyer $i \in \{1,\ldots,n\}$ by V_i, so that V_i is a random variable that takes nonnegative real values. We assume that all of the random variables $V_1,\ldots,V_n$ are independent. We also assume that $V_1,\ldots,V_n$ are identically distributed, with a continuous density function. That is, there exists some continuous function $f\colon \mathbf{R}\to[0,\infty)$ with $\int_{-\infty}^{\infty} f(x)dx = 1$ such that: for each $i \in \{1,\ldots,n\}$, for each $t \in \mathbf{R}$, the probability that $V_i \leq t$ is equal to $\int_{-\infty}^{t} f(x)dx$. We also assume that all buyers are **risk-neutral**, so that each buyer seeks to maximize their expected profits.

Finally, we assume that all of the above assumptions are **common knowledge**. That is, every player knows the above assumptions; every player knows that every player knows the above assumptions; every player knows that every player knows that every player knows the above assumptions; etc.

Under the above assumptions, a **pure strategy** for Player $i \in \{1,\ldots,n\}$ is a function $\beta_i\colon [0,1]\to[0,\infty)$. So, if Player i has a private value of V_i, he will make a bid of $\beta_i(V_i)$ in the auction. (We will not discuss mixed strategies in auctions.)

Given the strategies $\beta = (\beta_1,\ldots,\beta_n)$, and given any $v \in [0,1]$, Player i has expected profit $P_i(\beta, v)$, if her private value is v. (If buyer i wins the auction, and if buyer i has private value v and bid b, then the profit of buyer i is $v - b$.) We say that a strategy β is an **equilibrium** if, given any $v \in [0,1]$, any $b \geq 0$, and any $i \in \{1,\ldots,n\}$,

$$P_i(\beta, v) \geq P_i((\beta_1,\ldots,\beta_{i-1}, b, \beta_{i+1},\ldots,\beta_n), v).$$

Let n be a positive integer. Let $x = (x_1,\ldots,x_n) \in \{-1,1\}^n$. Let $f,g\colon \{-1,1\}^n \to \mathbf{R}$. For any subset $S \subseteq \{1,\ldots,n\}$, define a function $W_S\colon \{-1,1\}^n \to \mathbf{R}$ by $W_S(x) := \prod_{i\in S} x_i$. Define also the inner product $\langle f,g\rangle := 2^{-n}\sum_{x\in\{-1,1\}^n} f(x)g(x)$. Any $f\colon \{-1,1\}^n \to \mathbf{R}$ can be expressed as $f(x) = \sum_{S\subseteq\{1,\ldots,n\}} \langle f, W_S\rangle W_S(x)$. For any $S \subseteq \{1,\ldots,n\}$, if we denote $\widehat{f}(S) := \langle f, W_S\rangle = 2^{-n}\sum_{y\in\{-1,1\}^n} f(y)W_S(y)$, then we have $f(x) = \sum_{S\subseteq\{1,\ldots,n\}} \widehat{f}(S)W_S(x)$.

The **noise stability** of f with parameter $\rho \in (-1,1)$ is defined to be $\sum_{S\subseteq\{1,\ldots,n\}} \rho^{|S|}|\widehat{f}(S)|^2$.

Let $\mathcal{A} = \{1,\ldots,k\}$. Let $(P_a)_{a\in\mathcal{A}}$ be a set of probability distributions. Let H be a positive integer or ∞. A **bandit** problem is a two-player game with incomplete information played over H rounds. For each round $0 \leq t \leq H$ of the game, the learner chooses an action $A_t \in \mathcal{A}$. The learner then obtains a reward R_t, where R_t is sampled from P_{A_t}.

Since the game occurs over many rounds, A_t is a function of the history $A_0,\ldots,A_{t-1}$, $R_0,\ldots,R_{t-1}$ at time t. A **policy** π is a function whose input is the history (at any time t) and whose output is an action in $\mathcal{A}$.

For any $a \in \mathcal{A}$, let μ_a be the expected value of a random variable with distribution P_a. The **expected regret** r_H at time H is:

$$H\max_{a\in\mathcal{A}} \mu_a - \sum_{t=0}^{H-1} \mathbf{E}R_t.$$

1. Label the following statements as TRUE or FALSE. If the statement is true, **EXPLAIN YOUR REASONING**. If the statement is false, **PROVIDE A COUNTEREXAMPLE AND/OR EXPLAIN YOUR REASONING**.

   (a) (3 points) Every two-player zero-sum game has an optimal strategy.

   TRUE FALSE (circle one)

   (b) (3 points) Suppose I have a polynomial time algorithm that, when given any two-player general sum game defined by two $n \times n$ integer-valued payoff matrices, could output an ESS, or determine that no ESS exists. (This algorithm has a run time that is polynomial in n and in the log of the largest integer in the payoff matrices.) Then P=NP. (Equivalently, it is NP-hard to: find an ESS or decide no ESS exists.)

   TRUE FALSE (circle one)

   (c) (3 points) In the game of chess, it is known that the first player has a winning strategy. That is, the first player can guarantee a win, regardless of what the second player does.

   TRUE FALSE (circle one)

(d) (3 points) Consider the problem of deciding if a general sum game has at least two Nash equilibria. This problem is NP-complete.

TRUE FALSE (circle one)

(e) (3 points) Suppose we have a bandit problem with rewards satisfying $0 \leq R_t \leq 1$ for all times $t \geq 0$. Assume that $k \leq H$. Then there is an algorithm for the bandit problem with expected regret at time H bounded by $100\sqrt{kH \log H}$.

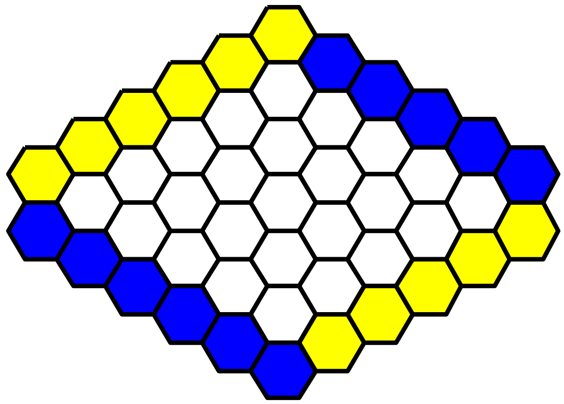
TRUE FALSE (circle one)

(f) (3 points) The Condorcet paradox no longer occurs if we consider an election between four candidates. That is, the Condorcet paradox only occurs in Condorcet elections between three candidates.

TRUE FALSE (circle one)

2. (10 points) Prove the following. On a standard Hex game board, the first player has a winning strategy. That is, the first player has a strategy that guarantees a win, regardless of what the second player does.

   (You may assume that exactly one person wins the game, so that the game never ends in a tie.)



: A Starting Position in Hex

3. (10 points) Let $m \geq 1$ be a positive integer.

   Show that Δ_m is convex and bounded.

   (You need to justify your answer.)

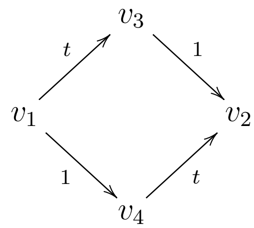
4. (10 points) Suppose we have a Vickrey auction, i.e. a sealed-bid second-price auction, with $n \geq 2$ bidders. Each bidder submits a bid to the seller (in a sealed envelope), and the winner of the auction is the highest bidder, and they pay the second-highest bid.

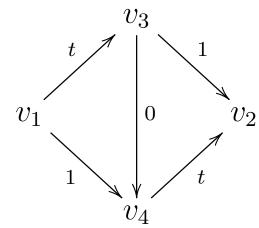
   - Explain why it is an equilibrium when each bidder bids their private value.
   - Explain why the total expected revenue of the seller is

$$n \int_{-\infty}^{\infty} F_Y(t)\mathbf{E}(Y \,|\, Y \leq t) f(t) dt,$$

     where $Y = \max(V_2, \ldots, V_n)$ is the maximum private value of $n-1$ bidders. (Hint: the expected payment of buyer 1 with private value t is $F_Y(t)\mathbf{E}(Y \,|\, Y \leq t)$ where $F_Y(t) := \mathbf{P}(Y \leq t)$ for all $t \in \mathbf{R}$.)

5. (10 points)





Consider a network with four vertices v_1, v_2, v_3, v_4 (cities) and four edges (roads): $(v_1, v_3), (v_3, v_2), (v_1, v_4), (v_4, v_2)$. Each edge has a cost which describes the time it takes for a driver to traverse that road. Suppose the edges have costs $t, 1, 1, t$, respectively. (A cost of t means: the cost is equal to the amount of traffic on that road.)

Suppose there is one unit of traffic, representing a large number of players. Each player wants to go from v_1 to v_2. Each player acts independently of each other player. And each player wants to minimize their travel time. Assume that every player is using the same strategy at equilibrium.

- Under the above assumptions, describe the unique Nash equilibrium for the players and the mean travel time of one player. Justify your answer.

Suppose now we add a short and fast (one way) highway from v_3 to v_4 with zero cost.

- Under the above assumptions, for the new highway system, describe the unique Nash equilibrium for the players and the mean travel time of one player. Justify your answer.
- What is the ratio between your current answer and your previous answer?

6. (10 points) Let $f\colon \{-1,1\}^n \to \{-1,1\}$.

- Show that the noise stability of f with parameter $0 < \rho < 1$ is at most 1.
- If $\sum_{x\in\{-1,1\}^n} f(x) = 0$, show that the noise stability of f with parameter $0 < \rho < 1$ is at most ρ.

7. (15 points) Suppose we have a two-player symmetric game with payoffs described by a matrix such as the following.

$$A = \begin{pmatrix} 4 & 3 & 2 & 5 & 6 \\ 3 & 1 & 8 & 9 & 1 \\ 7 & 0 & 7 & 0 & 7 \\ 1 & 3 & 3 & 2 & 1 \\ 0 & 1 & 2 & 9 & 1 \end{pmatrix}$$

Describe in detail an algorithm that outputs a Nash equilibrium of this game.

Make sure to justify why your algorithm outputs a Nash equilibrium.

Also give a bound on the run time of the algorithm.

Hint: you can freely use that, if $(\widetilde{x}, \widetilde{y}) \in \Delta_m \times \Delta_n$ is a Nash equilibrium and if

$$I := \{1 \leq i \leq m : \widetilde{x}_i > 0\}, \qquad J := \{1 \leq j \leq n : \widetilde{y}_j > 0\}. \qquad (*)$$

then with payoff matrices A, B respectively, we have

$$\max_{i=1,\ldots,m} (A\widetilde{y})_i = (A\widetilde{y})_i, \quad \forall\, i \in I. \qquad \text{and} \qquad \max_{j=1,\ldots,n} (\widetilde{x}^T B)_j = (\widetilde{x}^T B)_j, \quad \forall\, j \in J.$$

8. (10 points) Let $X_1, \ldots, X_n$ be real-valued i.i.d. (independent identically distributed) random variables. Assume that

$$\mathbf{E}e^{\alpha X_1} \leq e^{\alpha^2/2}, \qquad \forall\, \alpha \in \mathbf{R}.$$

Assume also that $\mathbf{E}X_1 = 0$. Prove that

$$\mathbf{P}\Big(\frac{1}{n}\sum_{i=1}^{n} X_i > t\Big) \leq e^{-nt^2/2}, \qquad \forall\, t > 0.$$

(Scratch paper)