499 Midterm 2 Solutions[1]

1. Question 1

TRUE/FALSE

(a) Let $K\subseteq\mathbf{R}^2$ be a convex set. Let $f\colon K\to K$ be continuous. Then f has a fixed point. That is, there exists $k\in K$ such that $f(k)=k$.

FALSE. Let $f(x,y)=(x+1,y)$ for all $(x,y)\in\mathbf{R}^2=:K$. Then f is continuous with no fixed point, since a fixed point (x,y) would satisfy $x+1=x$ which has no solution.

(b)There exists a symmetric two-person general-sum game such that all of its Nash equilibria are not symmetric.

FALSE. Every symmetric game has at least one symmetric Nash equilibrium, a corollary of Nash's Theorem.

(c) Every two-player general sum game has at least two correlated equilibria.

FALSE. The Prisoner's Dilemma has only one correlated equilibrium

(d) Any correlated equilibrium is a convex combination of Nash equilibria.

FALSE. In the game of chicken, we showed there is a correlated equilibrium which is not a convex combination of Nash equilibria.

2. Question 2

Recall the prisoner's dilemma, which has the following payoffs.

Prisoner I \ Prisoner II	silent	confess
silent	$(-1,-1)$	$(-10,0)$
confess	$(0,-10)$	$(-8,-8)$

Find all Nash equilibria for this game.

Solution. As shown in the notes, it follows from a domination argument that $x=(0,1)$ and $y=(0,1)$ is the only Nash equilibrium for this game.

3. Question 3

(a) Give an example of a convex and bounded subset K of Euclidean space, and give an example of a continuous function $f\colon K\to K$ such that f has no fixed point. *Solution.* Let $K=(0,1)$ and let $f(x)=x/2$ for all $x\in\mathbf{R}$. Then K is convex and bounded, and f is continuous, but f has no fixed points, since $f(x)=x$ would say $x/2=x$, i.e. $1/2=1$ (since $x>0$), which can never be satisfied.

(b) Give an example of a function $f\colon[0,1]\to[0,1]$ such that f has no fixed point.

Solution. For any $x\in[0,1)$, let $f(x)=1$, and let $f(1)=0$. Then f has no fixed point, since $f(x)=x$ is never satisfied, by definition of f.

4. Question 4

Recall that the game of Rock-Paper-Scissors is defined by the payoff matrices

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}, \quad B = A^T.$$

Then the game is symmetric. (And also, note that $A+B=0$, so that the game is a zero-sum game.)

You may assume that $(1/3, 1/3, 1/3)$ is a (symmetric) Nash equilibrium.

Show that this Nash equilibrium is **not** evolutionarily stable.

Solution. In order to see that $\vec{x} = (1/3, 1/3, 1/3)$ is not an evolutionarily stable equilibrium, we must find a different strategy $\vec{w}$ such that

$$\vec{w}^T A\vec{x} = \vec{x}^T A\vec{x} \quad \text{and} \quad \vec{w}^T A\vec{w} \geq \vec{x}^T A\vec{w}.$$

Since $A\vec{x} = (0,0,0)$ and $\vec{x}^T A = (0 \quad 0 \quad 0)$, it suffices to find $\vec{w}$ satisfying $\vec{w}^T A\vec{w} = 0$. To this end, we observe that any pure strategy $\vec{w}$ satisfies $\vec{w}^T A\vec{w} = 0$, as the payoff to both players is 0 if they both choose the same move. For example, $\vec{w} = (1,0,0)$ works.

5. Question 5

Define $v \colon 2^{\{1,2,3\}} \to \mathbf{R}$ so that $v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = v(\{1,2,3\}) = 1$, whereas $v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\emptyset) = 0$.

Arguing directly using **the axioms for the Shapley value**, compute all of the Shapley values of v.

Solution. We first claim that $\phi_1(v) = \phi_2(v) = \phi_3(v)$. This will follow by an application of Axiom (i). We need to check the assumption of Axiom (i) holds for all eight subsets S of $\{1,2,3\}$. When $S = \emptyset$, it is given that $v(\{1\}) = v(\{2\}) = v(\{3\})$. When $S = \{1\}$, we know $v(\{1,2\}) = v(\{1,3\})$; similarly the assumption of Axiom (i) holds for $S = \{2\}$ and $S = \{3\}$. The remaining assumptions of Axiom (i) hold vacuously (in the case that $|S| \geq 2$). We conclude that Axiom (i) tells us $\phi_1(v) = \phi_2(v) = \phi_3(v)$. Now, from Axiom (iii), $\phi_1(v) + \phi_2(v) + \phi_3(v) = v(\{1,2,3\}) = 1$. In conclusion, $\phi_1(v) = \phi_2(v) = \phi_3(v) = 1/3$.