Math 499, Spring 2025, USC  Instructor: Steven Heilman

Name: ______________ USC ID: ____________ Date: __________

Signature: ______________. Discussion Section: ______________
(By signing here, I certify that I have taken this test while refraining from cheating.)

# Exam 2

This exam contains 8 pages (including this cover page) and 5 problems. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- You have 50 minutes to complete the exam, starting at the beginning of class.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Do not write in the table to the right. Good luck![a]

| Problem | Points | Score |
|---|---|---|
| 1 | 8 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| Total: | 48 | |

**Reference sheet**

Below are some definitions that may be relevant.

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$$\Delta_m := \{x = (x_1, \ldots, x_m) \in \mathbf{R}^m \colon \sum_{i=1}^{m} x_i = 1,\, x_i \geq 0,\ \forall\, 1 \leq i \leq m\}.$$

Let $m, n$ be positive integers. Suppose we have a two-player general sum game with $m \times n$ payoff matrices. Let $A$ be the payoff matrix for player $I$ and let $B$ be the payoff matrix for player $II$. A pair of vectors $(\widetilde{x}, \widetilde{y})$ with $\widetilde{x} \in \Delta_m$ and $\widetilde{y} \in \Delta_n$ is a **Nash equilibrium** if

$$\widetilde{x}^T A \widetilde{y} \geq x A \widetilde{y}, \quad \forall\, x \in \Delta_m, \qquad \widetilde{x}^T B \widetilde{y} \geq \widetilde{x} B y, \quad \forall\, y \in \Delta_n.$$

A joint distribution of strategies is an $m \times n$ matrix $z = (z_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ such that $z_{ij} \geq 0$ for all $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$, and such that $\sum_{i=1}^{m} \sum_{j=1}^{n} z_{ij} = 1$. We say $z$ is a **correlated equilibrium** if

$$\sum_{j=1}^{n} z_{ij} a_{ij} \geq \sum_{j=1}^{n} z_{ij} a_{kj}, \qquad \forall\, i \in \{1, \ldots, m\}, \forall\, k \in \{1, \ldots, m\}.$$

$$\sum_{i=1}^{m} z_{ij} b_{ij} \geq \sum_{i=1}^{m} z_{ij} b_{ik}, \qquad \forall\, j \in \{1, \ldots, n\}, \forall\, k \in \{1, \ldots, n\}.$$

Suppose we have a two-player symmetric game (so that the payoff matrix for player $I$ is $A$, the payoff matrix for player $II$ is $B$, and with $A = B^T$). Assume that $A, B$ are $n \times n$ matrices. A mixed strategy $x \in \Delta_n$ is said to be an **evolutionarily stable strategy** if, for any pure strategy $w$, we have

$$w^T A x \leq x^T A x,$$
$$\text{If } w^T A x = x^T A x, \text{ then } w^T A w < x^T A w.$$

Suppose we have a game with $n$ players together with a characteristic function $v \colon 2^{\{1, \ldots, n\}} \to \mathbf{R}$. For each $i \in \{1, \ldots, n\}$, we define the **Shapley value** $\phi_i(v) \in \mathbf{R}$ to be any set of real numbers satisfying the following four axioms:

(i) (Symmetry) If for some $i, j \in \{1, \ldots, n\}$ we have $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq \{1, \ldots, n\}$ with $i, j \notin S$, then $\phi_i(v) = \phi_j(v)$.

(ii) (No power/ no value) If for some $i \in \{1, \ldots, n\}$ we have $v(S \cup \{i\}) = v(S)$ for all $S \subseteq \{1, \ldots, n\}$, then $\phi_i(v) = 0$.

(iii) (Additivity) If $u$ is any other characteristic function, then $\phi_i(v + u) = \phi_i(v) + \phi_i(u)$, for all $i \in \{1, \ldots, n\}$.

(iv) (Efficiency) $\sum_{i=1}^{n} \phi_i(v) = v(\{1, \ldots, n\})$.

1. Label the following statements as TRUE or FALSE. If the statement is true, **EXPLAIN YOUR REASONING**. If the statement is false, **PROVIDE A COUNTEREXAMPLE AND EXPLAIN YOUR REASONING**. Unlike other questions on this exam, you may cite homework problems to complete this part of the exam.

   (a) (2 points) Let $K \subseteq \mathbf{R}^2$ be a convex set. Let $f\colon K \to K$ be continuous. Then $f$ has a fixed point. That is, there exists $k \in K$ such that $f(k) = k$.

   TRUE FALSE (circle one)

   (b) (2 points) There exists a symmetric two-person general-sum game such that all of its Nash equilibria are not symmetric.

   TRUE FALSE (circle one)

   (c) (2 points) Every two-player general sum game has at least two correlated equilibria.

   TRUE FALSE (circle one)

   (d) (2 points) Any correlated equilibrium is a convex combination of Nash equilibria.

   TRUE FALSE (circle one)

2. (10 points) Recall the prisoner's dilemma, which has the following payoffs.

| | | Prisoner $II$ | |
|---|---|---|---|
| | | silent | confess |
| Prisoner $I$ | silent | $(-1,-1)$ | $(-10,0)$ |
| | confess | $(0,-10)$ | $(-8,-8)$ |

Find all Nash equilibria for this game. JUSTIFY YOUR ANSWER.

[this example was done in class]

3. In all parts of this problem, JUSTIFY YOUR ANSWER.

   [these examples were discussed in class]

   (a) (5 points) Give an example of a convex and bounded subset $K$ of Euclidean space, and give an example of a continuous function $f\colon K \to K$ such that $f$ has no fixed point.

   (b) (5 points) Give an example of a function $f\colon [0,1] \to [0,1]$ such that $f$ has no fixed point.

4. (10 points) Recall that the game of Rock-Paper-Scissors is defined by the payoff matrices

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}, \quad B = A^T.$$

Then the game is symmetric. (And also, note that $A + B = 0$, so that the game is a zero-sum game.)

You may assume that $(1/3, 1/3, 1/3)$ is a (symmetric) Nash equilibrium.

Show that this Nash equilibrium is **not** evolutionarily stable.

[this was a repeated homework exercise]

5. (10 points) Define $v\colon 2^{\{1,2,3\}} \to \mathbf{R}$ so that $v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = v(\{1,2,3\}) = 1$, whereas $v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\emptyset) = 0$.

   Arguing directly using **the axioms for the Shapley value**, compute all of the Shapley values of $v$. JUSTIFY YOUR ANSWER.

   [this example was done in class]

(Scratch paper)