499 Midterm 1 Solutions[1]

1. Question 1

TRUE/FALSE

(a) In any impartial combinatorial game under normal play, any game position lies in $\mathbf{N}$ or in $\mathbf{P}$.

FALSE. If the game is not progressively bounded, this might not be true. Take for example a modified version of Nim where a player can elect to not remove any chips, and the game position is two piles of chips with one chip in each pile. Then the game can go on forever, i.e. this game position is not in $\mathbf{N}$ or $\mathbf{P}$.

(b) Let A be a real 10×10 matrix. Then

$$\max_{x\in\Delta_{10}} \min_{y\in\Delta_{10}} x^T A y = \min_{y\in\Delta_{10}} \max_{x\in\Delta_{10}} x^T A y.$$

TRUE. This follows from Von Neumann's Minimax Theorem.

(c) Let A be a real 10×10 matrix. Let $(\widetilde{x}, \widetilde{y})$ be optimal strategies for the two person zero sum game with payoff matrix A. Then

$$\widetilde{x}^T A \widetilde{y} = \max_{x\in\Delta_{10}} \min_{y\in\Delta_{10}} x^T A y.$$

TRUE. We showed this in a remark after the proof of the Minimax Theorem.

(d) Optimal strategies are unique. That is, for any positive integers m, n, and for any real $m \times n$ matrix A, there is at most one pair of optimal strategies $(\widetilde{x}, \widetilde{y})$ for the two person zero sum game with payoff matrix A, where $\widetilde{x} \in \Delta_m$ and $\widetilde{y} \in \Delta_n$.

FALSE. We showed in class an example where optimal strategies were not unique. Here is another such example with $m = n$. If A is the identity matrix, then $x^T A y = x^T y$, and the Cauchy-Schwarz inequality implies that $x^T y \leq \|x\| \, \|y\| \leq 1$, since $x, y \in \Delta_m$. Equality occurs when $x = y$ is a standard basis vector (i.e. a vector with a single 1 entry and the rest zeros.) So, optimal strategies are not unique in this case when $m \geq 2$.

2. Question 2

Consider the game of Nim, where the game starts with four piles of chips. These piles have $1, 5, 3$ and 15 chips, respectively. Which player has a winning strategy from this position, the first player, or the second? Describe a winning first move.

Solution. In binary, the piles have 0001, 0111, 0010 and 1111 chips. So, the nim sum of the game position is $1011 \neq 0$. From Bouton's Theorem (Theorem 2.26 in the notes), the first player therefore has a winning strategy, and a winning first move is to force the nim sum to be zero. Such a move can be achieved by removing 11 chips from the pile that has 15 chips. The resulting game position will be $(1, 7, 2, 4)$. Or, in binary, the piles have 0001, 0111, 0010 and 0100 chips. So, the nim sum of this game position is zero.

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3. Question 3

Let $n \geq 2$ be an integer. Prove that Δ_n is convex and bounded.

Solution. Let $x, y \in \Delta_n$ and let $t \in [0, 1]$. By definition of Δ_n, $\sum_{i=1}^n x_i = 1$ and $x_i \geq 0$ for all $1 \leq i \leq n$, and $\sum_{i=1}^n y_i = 1$ and $y_i \geq 0$ for all $1 \leq i \leq n$. Since $t + (1-t) = 1$, we then have

$$\sum_{i=1}^n tx_i + \sum_{i=1}^n (1-t)y_i = t(1) + (1-t)1 = t + (1-t) = 1.$$

That is, $tx + (1-t)y$ satisfies $\sum_{i=1}^n (tx + (1-t)y)_i = 1$. Since $t, 1-t \geq 0$, and $x_i, y_i \geq 0$ for all $1 \leq i \leq n$, we have $tx_i + (1-t)y_i \geq 0$ for all $1 \leq i \leq n$. Then is $(tx + (1-t)y)_i \geq 0$ for all $1 \leq i \leq n$. In summary, we have shown that $tx + (1-t)y \in \Delta_n$. That is, Δ_n is convex.

For boundedness, we use $0 \leq x_i \leq 1$, so that $x_i^2 \leq 1$ for all $1 \leq i \leq n$ to get

$$\|x\| = \sqrt{\sum_{i=1}^n x_i^2} \leq \sqrt{\sum_{i=1}^n 1} = \sqrt{n}.$$

So, we may choose $r := \sqrt{n}$ to see that Δ_n is bounded. We could also use $r = 1$, since $0 \leq x_i \leq 1$ implies $0 \leq x_i^2 \leq x_i$ for all $1 \leq i \leq n$, so

$$\|x\| = \sqrt{\sum_{i=1}^n x_i^2} \leq \sqrt{\sum_{i=1}^n x_i} = \sqrt{1} = 1,$$

using $x \in \Delta_n$ in the penultimate equality.

4. Question 4

Find the value of the two-person zero-sum game described by the payoff matrix:

		Player *II*	
		W	X
Player *I*	Y	0	1
	Z	3	0

Solution. Let P denote the payoff matrix. Write $x = (a, 1-a)$ and $y = (b, 1-b)$ where $a, b \in [0, 1]$. Then $x^T P y = (a, 1-a)^T (1-b, 3b) = a(1-b) + 3(1-a)b = a - 4ab + 3b$. So,

$$\max_{x \in \Delta_2} \min_{y \in \Delta_2} x^T P y = \max_{a \in [0,1]} \min_{b \in [0,1]} (a - 4ab + 3b).$$

The minimum $\min_{b \in [0,1]}(a - 4ab + 3b)$ occurs at $b = 0$ or $b = 1$ since the function $a - 4ab + 3b$ is linear in $b \in [0, 1]$, so that

$$\min_{b \in [0,1]} (a - 4ab + 3b) = \min(a, a - 4a + 3) = \min(a, -3a + 3).$$

This function is piecewise linear on $a \in [0, 1]$ with maximum value occurring when $a = -3a + 3$, i.e. when $4a = 3$, i.e. $a = 3/4$, taking the value $3/4$. In summary,

$$\max_{x \in \Delta_2} \min_{y \in \Delta_2} x^T P y = \max_{a \in [0,1]} \min(a, -3a + 3) = 3/4.$$

That is, the value of the game is $3/4$.

5. Question 5

Find the value of the two-person zero-sum game described by the payoff matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 5 & 8 & 10 & 1 \end{pmatrix}$$

Describe optimal strategies for this game.

Solution. The second row dominates the first row, so we can ignore the first row for the purpose of computing the value of the game. That is, we can equivalently compute the value of the matrix

$$\begin{pmatrix} 5 & 6 & 7 & 8 \\ 5 & 8 & 10 & 1 \end{pmatrix}$$

The first column is dominated by the second and third column, so the second and third column can be ignored. That is, we have reduced to computing the value of the matrix

$$\begin{pmatrix} 5 & 8 \\ 5 & 1 \end{pmatrix}.$$

The first row dominates the second, so we have reduced to computing the value of the matrix

$$\begin{pmatrix} 5 & 8 \end{pmatrix}.$$

Finally, the first row is dominated by the second, so we have reduced to computing the value of the matrix

$$\begin{pmatrix} 5 \end{pmatrix}.$$

Since the game is now one-by-one, the value of the game is 5. The optimal strategies correspond to the strategies found in the domination argument, where we found the second row and first column of the payoff matrix was dominant. That is, optimal strategies are $x = (0, 1, 0)$ and $y = (1, 0, 0, 0)$.