Math 499, Spring 2025, USC — Instructor: Steven Heilman

Name: ______________ USC ID: ______________ Date: ______________

Signature: ______________. Discussion Section: ______________
(By signing here, I certify that I have taken this test while refraining from cheating.)

# Exam 1

This exam contains 8 pages (including this cover page) and 5 problems. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- You have 50 minutes to complete the exam, starting at the beginning of class.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Do not write in the table to the right. Good luck![a]

| Problem | Points | Score |
|---|---|---|
| 1 | 8 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| Total: | 48 | |

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## Reference sheet

Below are some definitions that may be relevant.

An **impartial combinatorial game** is a combinatorial game with two players who both have the same set of legal moves. A **terminal position** is a position from which there are no legal moves. Every non-terminal position has at least one legal move. Under **normal play**, the player who moves to the terminal position wins.

For any impartial combinatorial game, let **N** (for "next") be the set of game positions such that the first player to move can guarantee a win. Let **P** denote the set of game positions such that *any* legal move leads to a position in **N**. We also let **P** contain all terminal positions.

Let $m$ be a positive integer.

$$\Delta_m := \{x = (x_1, \ldots, x_m) \in \mathbf{R}^m \colon \sum_{i=1}^m x_i = 1,\, x_i \geq 0,\ \forall\, 1 \leq i \leq m\}.$$

Let $m, n$ be positive integers. Let $A$ be an $m \times n$ real matrix. Then the **value** of the two-person zero-sum game with payoff matrix $A$ is

$$\max_{x \in \Delta_m} \min_{y \in \Delta_n} x^T A y.$$

A mixed strategy $\widetilde{x} \in \Delta_m$ is **optimal for player** $I$ if

$$\min_{y \in \Delta_n} \widetilde{x}^T A y = \max_{x \in \Delta_m} \min_{y \in \Delta_n} x^T A y.$$

A mixed strategy $\widetilde{y} \in \Delta_n$ is **optimal for player** $II$ if

$$\max_{x \in \Delta_m} x^T A \widetilde{y} = \min_{y \in \Delta_n} \max_{x \in \Delta_m} x^T A y.$$

We say the pair $(\widetilde{x}, \widetilde{y})$ are **optimal strategies** for the payoff matrix $A$ if $\widetilde{x} \in \Delta_m$ is optimal for player $I$ and $\widetilde{y} \in \Delta_n$ is optimal for player $II$.

For any $x = (x_1, \ldots, x_m) \in \mathbf{R}^m$, we denote

$$||x|| := \Big(\sum_{i=1}^m x_i^2\Big)^{1/2}.$$

1. Label the following statements as TRUE or FALSE. If the statement is true, **explain your reasoning**. If the statement is false, **provide a counterexample and explain your reasoning**.

   (a) (2 points) In any impartial combinatorial game under normal play, any game position lies in **N** or in **P**.

   TRUE FALSE (circle one)

   (b) (2 points) Let $A$ be a real $10 \times 10$ matrix. Then

   $$\max_{x \in \Delta_{10}} \min_{y \in \Delta_{10}} x^T A y = \min_{y \in \Delta_{10}} \max_{x \in \Delta_{10}} x^T A y.$$

   TRUE FALSE (circle one)

   (c) (2 points) Let $A$ be a real $10 \times 10$ matrix. Let $(\widetilde{x}, \widetilde{y})$ be optimal strategies for the two person zero sum game with payoff matrix $A$. Then

   $$\widetilde{x}^T A \widetilde{y} = \max_{x \in \Delta_{10}} \min_{y \in \Delta_{10}} x^T A y.$$

   TRUE FALSE (circle one)

   (d) (2 points) Optimal strategies are unique. That is, for any positive integers $m, n$, and for any real $m \times n$ matrix $A$, there is at most one pair of optimal strategies $(\widetilde{x}, \widetilde{y})$ for the two person zero sum game with payoff matrix $A$, where $\widetilde{x} \in \Delta_m$ and $\widetilde{y} \in \Delta_n$.

   TRUE FALSE (circle one)

2. (10 points) Consider the game of Nim, where the game starts with four piles of chips. These piles have $1, 7, 2$ and 15 chips, respectively. Which player has a winning strategy from this position, the first player, or the second? Describe a winning first move.

   [This was a modified practice exam question.]

3. (10 points) Let $n \geq 2$ be an integer. Prove that $\Delta_n$ is convex and bounded.

   (Recall a set $K \subseteq \mathbf{R}^n$ is convex if, for any $x, y \in K$ and for any $t \in [0,1]$, we have $tx + (1-t)y \in K$.)

   (Recall a set $K \subseteq \mathbf{R}^n$ is bounded if there exists $r > 0$ such that $||x|| \leq r$ for all $x \in K$.)

   [This was a homework question.]

4. (10 points) Find the value of the two-person zero-sum game described by the payoff matrix:

| | | Player *II* | |
|---|---|---|---|
| | | W | X |
| Player *I* | Y | 0 | 1 |
| | Z | 3 | 0 |

[This was a modified homework question.]

5. (10 points) Find the value of the two-person zero-sum game described by the payoff matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 5 & 8 & 10 & 1 \end{pmatrix}$$

Describe optimal strategies for this game.

[This was a modified homework question]

(Scratch paper)